29. Solve the following model of optimal subdivision of a cable of length 20 units into 4 parts such that the product of their lengths is maximized using dynamic programming technique.

Max. $z = p_1 \, p_2 \, p_3 \, p_4$, subject to $p_1 + p_2 + p_3 + p_4 = 20$; $p_1, p_2, p_3, p_4 \geq 0$. **[JNTU (B. Tech.) 2003]**

SELF-EXAMINATION REVIEW QUESTIONS

1. Define the following terms in dynamic programming :
   (i) Stage (ii) State (iii) State variable (iv) Decision variable
   (v) Immediate return (vi) Optimal return (vii) State transformation function.
2. State Bellman's principle of optimality and explain by an illustrative example how it can be used to sovle multistage decision problem.
3. Write short notes on :
   (i) Dynamic programming (ii) Applications of dynamic programming
   (iii) Warehousing problem in dynamic proggramming.
4. (a) Describe the deterministic dynamic programming.
   (b) Discuss dynamic programming with suitable examples.
   (c) Define standard warehouse problem. Outline a procedure to solve it.
5. Explain the forward index method of dynamic programming for solving any problem when the objective function can be decomposed into : (i) additionally, (ii) multiplicatively.
6. What are the essential characteristics of dynamic programming problem ?
7. Discuss the basic features which characterize the dynamic programming problem.
8. Explain clearly the technique of dynamic programming in solving programming problems. Explain the nature of the computational method of dynamic programming in solving max $\Sigma f_j(x_j)$ subject to
   $$\Sigma \, a_j \, x_j = b, \; a_j > 0, \; x_j \geq 0, \; j = 1, 2, \ldots, n$$
   all x_j's integers.
9. Use dynamic programming to indicate how you would arrive at an optimal distribution of crates to the stores so as to maximize the total expected profit.
10. How dynamic programming can be used to solve a man-power loading problem ? **[JNTU (B. Tech.) 2003]**
11. What is dynamic programming ? In what areas of management can it be applied successfully ?
12. Show that $\max_{d_2} \psi_2 \, [f_2(s_2, d_2) \; o \; \min_{d_1} f_1(s_2, d_2, d_1)] = \max_{d_2, d_1} \psi_2 \, [f_2(s_2, d_2) \; o \; f_1(s_2, d_2, d_1)]$
    where ψ_2 is a monotonic non-increasing function of f_1 for every f_2
13. Discuss the stochastic gold-mining problem and prove the existance and uniqueness theorem.
14. Explain the curse of dimensionality. **[Meerut 2002]**
15. (a) State Bellman's principle of optimality. A positive quantity C is to be divided into n parts is such a way that the product of n parts is to be maximum. Without solving the problem completely as a dynamic programming problem, indicate clearly the stage at which the application of the principle of optimality becomes essential for proceeding further.
    (b) The above problem can be solved directly by using the method of Lagrangian multipliers. Explain, why the dynamic programming approach is better then the classical calculas approach in such problems. **[Meerut (OR) 2003]**

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APPENDIX

Theorem. *If the symbols have the same meaning as given in* ***Theorem 27.4 (Unit 5)****, then*

$$\frac{\partial f(\bar{\mathbf{x}})}{\partial b_i} = \bar{\lambda}_i$$

Proof. From the chain rule of differential calculus we have

$$\frac{\partial f(\bar{\mathbf{x}})}{\partial b_i} = \sum_{j=1}^{n} \frac{\partial f(\bar{\mathbf{x}})}{\partial b_i} \frac{\partial x_j}{\partial b_i}\Bigg|_{\mathbf{x}=\bar{\mathbf{x}}}$$

$$\frac{\partial g_k(\mathbf{x}, \mu_k)}{\partial b_i}\Bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mu_k=\bar{\mu}_k}} = \sum_{j=1}^{n} \frac{\partial g_k(\mathbf{x}, \mu_k)}{\partial x_j}\Bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mu_k=\bar{\mu}_k}} = \frac{\partial x_j}{\partial b_i} = y_{ik}$$

where

$$y_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Multiplying y_{ik} by λ_k and summing over all values of k gives us

$$\sum_{k=1}^{m} \bar{\lambda}_k y_{ik} = \sum_{k=1}^{m} \bar{\lambda}_k \sum_{j=1}^{n} \frac{\partial g_k(\mathbf{x}, \mu_k)}{\partial x_j}\Bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mu_k=\bar{\mu}_k}} \frac{\partial x_j}{\partial b_i}$$

The expression given above equals λ_1, since $y_{ik} = 0$ unless $k = i$. Therefore,

$$\bar{\lambda}_i = \sum_{k=1}^{m} \bar{\lambda}_k \sum_{j=1}^{n} \frac{\partial g_k(\mathbf{x}, \mu_k)}{\partial x_j}\Bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mu_k=\bar{\mu}_k}} \frac{\partial x_j}{\partial b_i}$$

Subtracting this result from $\frac{\partial f(\bar{\mathbf{x}})}{\partial b_i}$; we obtain

$$\frac{\partial f(\bar{\mathbf{x}})}{\partial b_i} - \bar{\lambda}_i = \sum_{j=1}^{n} \left[\frac{\partial f(\mathbf{x})}{\partial x_j} - \sum_{k=1}^{m} \bar{\lambda}_k \frac{\partial g_k(\mathbf{x}, \mu_k)}{\partial x_j}\Bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mu_k=\bar{\mu}_k}} \frac{\partial x_j}{\partial b_i} \right]$$

However, the right hand side vanishes, because the terms in brackets must equal zero to satisfy the necessary conditions. Thus,

$$\frac{\partial f(\bar{\mathbf{x}})}{\partial b_i} = +\bar{\lambda}_i.$$

This proves the theorem.

Remark. This theorem gives us a quantitative measure of the check in the optimal value of the objective function with respect to changes in the constraints. Consequently, the λ_i have been called sensitivity co-efficients as well as dual variables, adjoint functions, and shadow prices [26].

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UNIT 6

CONTAINING :

Appendix A. THEORY OF SIMPLEX METHOD

Appendix B. A NEW METHOD FOR INITIAL SOLUTION OF TRANSPORTATION PROBLEM

Appendix C. NUMERICAL TABLES

Appendix D. SELECTED REFERENCES

APPENDIX-A

THEORY OF SIMPLEX METHOD

A-0. INTRODUCTION

Sometimes we are interested to learn the complete derivation of *simplex algorithm.* For this purpose, all steps of *simplex method* have been fully developed by proving a number of *theorems* in an orderly manner.

For more details (with numerical examples), author's publication '***Linear Programming & Theory of Games***' may be consulted.

A-1. FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

Theorem (A-1). (Reduction of F.S. to B.F.S.). *If a linear programming problem max* $z = \mathbf{CX}$ *subject to* $A\mathbf{X} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, *has at least one feasible solution, then it has at least one basic feasible solution.*

Proof. Let $A\mathbf{X} = \mathbf{b}$ be the linear system in standard form and A be $m \times (m+n)$ matrix. Consider an arbitrary feasible solution

$$\mathbf{X}^{(0)} = (x_1, x_2, ..., x_{n+m}). \quad ...(A.1)$$

Also, suppose that the variables have been numbered such that those variables which have *positive* values are k first ones $(k \leq m+n)$, the remaining $(m+n-k)$ variables having the value zero. Thus, (A·1) can be expressed as

$$\mathbf{X}^{(0)} = (x_1, x_2, ..., x_k, \overbrace{0, 0 .., 0}^{m+n-k}). \quad ...(A.2)$$

Let $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_k$, be the first k columns of A (associated with the non-zero variables $x_1, x_2, .., x_k$, respectively. Then by hypothesis,

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_k\mathbf{a}_k = \mathbf{b} \text{ or } \sum_{j=1}^{k} x_j \mathbf{a}_j = \mathbf{b} \quad ...(A.3)$$

But, only two possibilities may arise : that is, the set of vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_k$ may be *either linearly independent* or *dependent.* Now consider these two cases individually.

Case 1. If $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_k$ ***are linearly independent.***

Any *feasible solution,* for which vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_k$ associated with non-zero variables $x_1, x_2, ..., x_k$, respectively, are linearly independent, is called a *basic feasible solution.* Hence first part of the theorem is true in this case, that is, the feasible solution $x_1, x_2, ..., x_k$; $x_{k+1} = 0, ..., x_{m+n} = 0$ is, by definition, a basic feasible solution. This solution is degenerate if $k < m$, and *non-degenerate* if $k = m$.

Case 2. If $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_k$ ***are linearly dependent.***

This is certainly the case when $k > m$. In this case, reduce the number of positive variables (step-by-step) until the columns associated with positive variables become linearly independent. From this feasible solution another feasible solution can be obtained.

Since $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_k$ are linearly dependent, then by definition of linear dependence, there exist scalars $\lambda_j, j = 1, 2, ..., k$ such that

$$\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + ... + \lambda_k\mathbf{a}_k = \mathbf{0} \text{ or } \sum_{j=1}^{k} \lambda_j \mathbf{a}_j = \mathbf{0} \quad ...(A.4)$$

implies the existence of at least one $\lambda_j \neq 0$.

Suppose at least one of the λ_j is positive, if it were not, we can multiply the equation (A.4) by -1.
Now, let

$$\nu = \max_{1 \le j \le k} (\lambda_j / x_j) \qquad \text{...(A.5)}$$

Obviously, $\nu > 0$ for $x_j > 0$ ($j = 1, 2, ..., k$) and at least one $\lambda_j > 0$.
Now, multiplying the equation (A.4) by $1/\nu$ and then subtracting from (A.3),

$$\sum_{j=1}^{k} x_j \mathbf{a}_j - \frac{1}{\nu} \sum_{j=1}^{k} \lambda_j \mathbf{a}_j = \mathbf{b} \text{ or } \sum_{j=1}^{k} \left(x_j - \frac{\lambda_j}{\nu} \right) \mathbf{a}_j = \mathbf{b}$$

which states that.

$$\hat{\mathbf{X}} = \left(x_1 - \frac{\lambda_1}{\nu}, x_2 - \frac{\lambda_2}{\nu}, ..., x_k - \frac{\lambda_k}{\nu}, \overbrace{0, 0 ..., 0}^{m+n-k} \right) \qquad \text{...(A.6)}$$

is a new *solution* of matrix equation $A\mathbf{x} = \mathbf{b}$. Also, from (A.5)

$$\nu \ge \frac{\lambda_j}{x_j} \text{ or } x_j \ge \frac{\lambda_j}{\nu} \text{ or } x_j - \frac{\lambda_j}{\nu} \ge 0, \ j = 1, 2, ..., k.$$

Thus, new solution $\hat{\mathbf{X}}$ also satisfies the non-negativity conditions.

Since $x_j - (\lambda_j/\nu) = 0$ for at least one j, $\hat{\mathbf{X}}$ given by (A.6) is a feasible solution which contains at the most $k-1$ non-zero variables. all other variables will be 0.

If the columns associated with positive variables are still *linearly dependent*, repeat the whole reduction procedure as described above. Eventually, derive a solution in which columns corresponding to positive variables are linearly independent (none of the $\mathbf{a}_j = \mathbf{0}$ and that a set containig a single non-null vector is always *linearly independent*).

Thus the theorem is completely proved.

Q. Prove that, if system $AX = \mathbf{b}$, of m linear equations in n unknowns ($m < n$) with rank (A) = m, has a feasible solution, then it has a basic feasible solution also. **[Meerut (LP) 98 BP]**

Theorem (A-2). *If a linear programming problem,*

$$\max.\ z = \mathbf{CX}, \text{ such that } A\mathbf{X} = \mathbf{b}, \mathbf{X} \ge \mathbf{0},$$

has at least an optimal feasible solution, then at least one basic feasible solution must be optimal. **[Meerut 90]**

Proof. Let

$$\mathbf{X}^{(0)} = (x_1, x_2, ..., x_k, \overbrace{0, 0 ..., 0}^{m+n-k}) \qquad \text{...(A.7)}$$

be an optimal feasible solution to the given linear programming problem which yields the optimum value

$$z^* = \sum_{j=1}^{k} c_j x_j \qquad \text{...(A.8)}$$

Now, proceeding as in *Theorem (A.1)*, we can reduce $\mathbf{X}^{(0)}$ to the new basic feasible solution

$$\hat{\mathbf{X}} = \left(x_1 - \frac{\lambda_1}{\nu}, x_2 - \frac{\lambda_2}{\nu}, ..., x_k - \frac{\lambda_k}{\nu}, \overbrace{0, 0 ..., 0}^{m+n-k} \right) \qquad \text{...(A.9)}$$

Now, to prove that $\hat{\mathbf{X}}$ *is also an optimum solution.*

The new value of the objective function corresponding to this solution $\hat{\mathbf{X}}$ will become

$$\hat{z} = \sum_{j=1}^{k} c_j \left(x_j - \frac{\lambda_j}{\nu} \right) = \sum_{j=1}^{k} c_j x_j - \frac{1}{\nu} \sum_{j=1}^{k} c_j \lambda_j$$

or

$$\hat{z} = z^* - \frac{1}{\nu} \sum_{j=1}^{k} c_j \lambda_j \qquad \left[\text{since } z^* = \sum_{j=1}^{k} c_j x_j \text{ (from eq.(A.8)}\right] \qquad \text{...(A.10)}$$

For optimality, $\hat{z}$ must be equal to z^*. Hence, $\hat{\mathbf{X}}$ will be optimal solution, if and only if,

$$\sum_{j=1}^{k} c_j \lambda_j = 0 \qquad \text{... (A.11)}$$

in equation (A.10). This can be proved by contradiction.

If possible, let us suppose that

$$\sum_{j=1}^{k} c_j \lambda_j \neq 0$$

Then, there will be two possibilities : (*i*) $\sum_{j=1}^{k} c_j \lambda_j > 0$, or (*ii*) $\sum_{j=1}^{k} c_j \lambda_j < 0$.

Now, in either of these two cases a real number (say, r) can be found out such that $r \sum_{j=1}^{k} c_j \lambda_j > 0$,

(in the first case, r will be positive, and in the second case r will be negative).

or

$$\sum_{j=1}^{k} c_j (r\lambda_j) > 0. \qquad ...(A.12)$$

Now adding $\sum_{j=1}^{k} c_j x_j$ to both sides of (A.12), we get

$$\sum_{j=1}^{k} c_j (r\lambda_j) + \sum_{j=1}^{k} c_j x_j > \sum_{j=1}^{k} c_j x_j \text{ or } \sum_{j=1}^{k} c_j (x_j + r\lambda_j) > z^*. \qquad ...(A.13)$$

Now, $(x_1 + r\lambda_1, x_2 + r\lambda_2, ..., x_k + r\lambda_k, \overbrace{0, 0..., 0}^{m+n-k})$ is also a solution for any value of r which can be observed by multiplying equation (A.4) by r and then adding to equation (A.3).

Furthermore, there *exist* an *infinite number of choices of r for which the solution*

$$(x_1 + r\lambda_1, x_2 + r\lambda_2, ..., x_k + r\lambda_k, \overbrace{0, 0 ..., 0}^{m+n-k})$$

satisfies the non-negativity restrictions as well :

In order to prove this statement and to satisfy the non-negativity restriction, we need

$$x_j + r\lambda_j \geq 0, j = 1, 2, ..., k$$

or

$$r\lambda_j \geq -x_j$$

or

$$r \geq -\frac{x_j}{\lambda_j}, \text{ if } \lambda_j > 0 \text{ or } r \leq -\frac{x_j}{\lambda_j} \text{ if } \lambda_j < 0, \text{ and } r \text{ is unrestricted if } \lambda_j = 0$$

Thus, if r is selected to satisfy the relationship :

$$\max_{\substack{j \\ (\lambda_j > 0)}} \left(-\frac{x_j}{\lambda_j} \right) \leq r \leq \min_{\substack{j \\ (\lambda_j < 0)}} \left(-\frac{x_j}{\lambda_j} \right), \qquad ...(A.14)$$

then $x_j + r\lambda_j \geq 0$ for $j = 1, 2, ..., k$. It may also be noted that if there is no j for which $\lambda_j > 0$, then there is no lower limit for r and if there is no j for which $\lambda_j < 0$, then there is no upper limit for r. Furthermore,

$$\max_{\substack{j \\ (\lambda_j > 0)}} \left(-\frac{x_j}{\lambda_j} \right) < 0 \text{ and } \min_{\substack{j \\ (\lambda_j < 0)}} \left(-\frac{x_j}{\lambda_j} \right) > 0.$$

This proves that when r lies in the non-empty interval given in (A-14), then the infinite number of solutions

$$(x_1 + r\lambda_1, x_2 + r\lambda_2, ..., x_k + r\lambda_k, \overbrace{0, 0 ..., 0}^{m+n-k})$$

satisfy the non-negativity restrictions as well.

Now, returning to (A.13), it may be concluded that left hand side $\sum_{j=1}^{k} c_j(x_j + r\lambda_j)$ yields the value of the objective function which is *strictly* greater than the greatest value of the objective function , which is not possible. This contradiction proves that (A.13) holds and hence $\hat{x}$ is optimal .

Alternative Proof. *By Theorem (A-1) if there is an optimal solution to an L. P. problem, max.* $z = \mathbf{CX}$ *subject to* $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, *then there is a basic feasible solution.* So a basic feasible solution must exist to the given problem.

Let $z_0 = \mathbf{C}_B \mathbf{X}_B$ with $\mathbf{X}_B = B^{-1}\mathbf{b}$, be a basic feasible solution to the problem.

Now to show that a basic feasible solution is optimal, we are required to prove $z_0 \geq z^*$.

The constraint $A\mathbf{X} = \mathbf{b}$ can be expressed as

$$\sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b} \qquad \text{...(A.15)}$$

Since any vector $\mathbf{a}_j \in A$ can be expressed as a linear combination of vectors in B,

$$\mathbf{a}_j = \sum_{i=1}^{m} x_{ij} \beta_i \qquad \text{...(A.16)}$$

From (A.15) and (A.16) we have

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \beta_i \right) x_j = \mathbf{b} \quad \text{or} \quad \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} x_j \right) \beta_i = \mathbf{b}$$

Since $\mathbf{X}_B = B^{-1}\mathbf{b}$ is the basic feasible solution to (A.15), $\sum_{i=1}^{m} x_{Bi} \beta_i = \mathbf{b}$.

Any vector can be uniquely expressed as a function of its basic vectors, therefore

$$\mathbf{X}_B = \sum_{i=1}^{m} x_{Bi} \beta_i .$$

The optimality condition is $z_j \geq c_j$ and $x_j \geq 0$. Therefore,

$$\sum_{j=1}^{n} x_j z_j \geq \sum_{j=1}^{n} x_j c_j = z^* \qquad \text{...(A.17)}$$

But, we have

$$\sum_{j=1}^{n} x_j z_j \geq \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} c_{Bi} x_{ij} \right)$$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_j x_{ij} \right) c_{Bi} = \sum_{i=1}^{m} x_{Bi} c_{Bi} = z_0 \qquad \text{...(A.18)}$$

Using (A.17) in (A.18), we have $z_0 \geq z^*$. Thus, basic feasible solution is optimal.

This completes the proof of the theorem.

Q. Given a **LP** problem in standard form, if it has an optimal solution show that at least one of the basic solutions will be optimal.

Note. 1. Sometimes Theorems $(A-1)$ and $(A-2)$ may be stated in combined form as follows:
"Show that if the linear programming problem: max. $z = \mathbf{c}\mathbf{x}$, *subject to* $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$
has feasible solution, then a least one of the basic feasible solutions will be optimal."

2. Theorems 1 and 2 of Sec. 2.7. in Unit 2 are the geometric analogue of Fundamental Theorems (A-1) and (A-2)

A-2. HOW TO DETERMINE IMPROVED FEASIBLE SOLUTION

Suppose a basic feasible solution to $A\mathbf{X} = \mathbf{b}$ given by

$$\mathbf{X}_B = (x_{B1}, x_{B2}, \ldots, x_{Bm}) = B^{-1}\mathbf{b} \qquad \text{...(A.19)}$$

and gives a value of the objective function $z = \mathbf{C}_B \mathbf{X}_B$

To develop a procedure for determining another basic feasible solution which gives a better value of z following theorems are given.

Theorem (A-3). (Replacement of a Basic Column.) *Given a non-degenerate basic feasible solution* $\mathbf{X}_B = B^{-1}\mathbf{b}$ *to* $A\mathbf{X} = \mathbf{b}$ *which yields a value for the objective function* $z = \mathbf{C}_B \mathbf{X}_B$. *If for any column* $\mathbf{a}_j$ *in A but not in* $\boldsymbol{B}$, $z_j - c_j < 0$ *and if at least one* $x_{ij} > 0$ $(i = 1, 2, \ldots m)$, *then a new basic feasible solution can be obtained by replacing one of the columns in B by* $\mathbf{a}_j$.

Proof. A new basic feasible solution is obtained by replacing one of the vectors (say $\mathbf{a}_j$) in A but not in $\boldsymbol{B}$ by some vecor in $\boldsymbol{B}$ (say β_r). Therefore,

$$\mathbf{a}_j \neq \beta_i \ (i = 1, 2, ..., m).$$

since $\mathbf{a}_j$ can be expressed as the linear combination of vectors in B, we have

$$\mathbf{a}_j = \sum_{i=1}^{m} x_{ij}\beta_i \ \text{ or } \ \mathbf{a}_j = x_{1j}\beta_1 + x_{2j}\beta_2 + ... + x_{rj}\beta_r + ... + x_{mj}\beta_m \qquad ...(A\text{-}20)$$

Now, using the replacement theorem $\mathbf{a}_j$ can replace β_r and still maintains the basis matrix, provided $x_{rj} \neq 0$.

Assuming $x_{rj} \neq 0$ from (A-20), $\mathbf{a}_j$ can be written as

$$\mathbf{a}_j = \sum_{i=1, i\neq r}^{m} x_{ij}\beta_i + x_{rj}\beta_r \qquad ...(A\text{-}21)$$

Solving the equation (A-21) for β_r,

$$\beta_r = \frac{1}{x_{rj}}\mathbf{a}_j - \sum_{i=1, i\neq r}^{m} \frac{x_{ij}}{x_{rj}}\beta_i \qquad ...(A.22)$$

Also $\qquad \mathbf{BX}_B = \mathbf{b}$ or $(\beta_1, \beta_2, ..., \beta_m)(x_{B1}, x_{B2}, ..., x_{Br}, ..., x_{Bm}) = \mathbf{b}$

or

$$(x_{B1}\beta_1, x_{B2}\beta_2, ..., x_{Bm}\beta_m) = \mathbf{b} \ \text{ or } \sum_{i=1, i\neq r}^{m} x_{Bi}\beta_i + x_{Br}\beta_r = \mathbf{b} \qquad ...(A.23)$$

Substituting the value of β_r from (A-22) in (A-23)

$$\sum_{i=1, i\neq r}^{m} x_{Bi}\beta_i + x_{Br}\left(\frac{1}{x_{rj}}\mathbf{a}_j - \sum_{i=1, i\neq r}^{m}\frac{x_{ij}}{x_{rj}}\beta_i\right) = \mathbf{b}$$

or

$$\sum_{i=1, i\neq r}^{m}\left(x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}}\right)\beta_i + \frac{x_{Br}}{x_{rj}}\mathbf{a}_j = \mathbf{b} \qquad ...(A\text{-}24)$$

or

$$\sum_{i=1, i\neq r}^{m} \hat{x}_{Bi}\beta_i + \hat{x}_{Br}\mathbf{a}_j = \mathbf{b} \qquad ...(A\text{-}25)$$

where

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}}, \ i = 1, 2, ..., m; i \neq r \qquad ...(A\text{-}26)$$

and

$$\hat{x}_{Br} = \frac{x_{Br}}{x_{rj}} \ \text{ for } i = r. \qquad ...(A\text{-}27)$$

Comparision of (A-25) with (A-23) shows that the new basic solution of $A\mathbf{X} = \mathbf{b}$ becomes

$$\hat{\mathbf{X}}_B = (\hat{x}_{Bi}, \hat{x}_{Br}), i = 1, 2, ..., m; i \neq r$$

$$= (\hat{x}_{B1}, \hat{x}_{B2}, ..., \hat{x}_{Br}, ..., \hat{x}_{Bm}) = \left(x_{B1} - x_{Br}\frac{x_{1j}}{x_{rj}}, x_{B2} - x_{Br}\frac{x_{2j}}{x_{rj}}, ..., \frac{x_{Br}}{x_{rj}}, ..., x_{Bm} - x_{Br}\frac{x_{mj}}{x_{rj}}\right)$$

and other non-basic components are zero.

For the new basic solution to be feasible, $\hat{x}_{Bi} \geq 0$, $i = 1, 2, ..., m$, that is, from (A-26) and (A-27)

$$x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}} \geq 0, \ i = 1, 2, ..., m, i \neq r \qquad ...(A.28)$$

$$\frac{x_{Br}}{x_{rj}} \geq 0., \ i = r \qquad ...(A.29)$$

From (A-28) $x_{rj} > 0$ since we start with a non-degenerate basic feasible solution, $x_{Bi} > 0$, $(i = 1, 2, ..., m)$. If $x_{rj} > 0$, and $x_{ij} \leq 0 \ (i \neq r)$, then (A-28, A-29) are satisfied. If $x_{rj} > 0$ and $x_{ij} > 0$, $(i \neq r)$, then equations (A-28, A-29) are satisfied only when

$$\frac{x_{Bi}}{x_{ij}} - \frac{x_{Br}}{x_{rj}} \geq 0 \ \text{ [dividing (A-28) by } x_{ij} > 0] \qquad ...(A.30)$$

or $$-\frac{x_{Br}}{x_{rj}} \geq -\frac{x_{Bi}}{x_{ij}} \text{ or } \frac{x_{Br}}{x_{rj}} \leq \frac{x_{Bi}}{x_{ij}} \text{ or } \frac{x_{Br}}{x_{rj}} = \min_i \left\{ \frac{x_{Bi}}{x_{ij}} \right\}$$

Thus, if r is selected such that

$$v = \frac{x_{Br}}{x_{rj}} = \min_i \left\{ \frac{x_{Bi}}{x_{ij}}, x_{ij} > 0 \right\} \qquad \text{...(A-31)}$$

then column β_r will be removed from basis matrix B to replace $\mathbf{a}_j$ so that the new basic solution will be feasible.

Hence the theorem is proved.

Note 1. A new non-singular matrix obtained from B by replacing β_r with $\mathbf{a}_j$, is denoted by $\hat{B} = (\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_m)$ where $\hat{\beta}_i = \beta_i$, $i \neq r$, and $\hat{\beta}_r = \mathbf{a}_j$.

2. If the minimum in (A-31) is not unique, the new basic solution will be degenerate. Because, in this case, the number of positive basic variables will become less than m.

Q. How would you proceed to change the basic feasible solution in case it is not optimal.

Theorem (A-4). *Assume that a non-degenerate basic feasible solution* $\mathbf{X}_B = B^{-1}\mathbf{b}$ *to* $A\mathbf{X} = \mathbf{b}$ *which yields a value for the objective function* $z = \mathbf{C}_B\mathbf{X}_B$. *Further suppose that a new basic feasible solution* $\hat{\mathbf{X}} = \hat{B}^{-1}\mathbf{b}$ *to* $A\mathbf{X} = \mathbf{b}$ *obtained by replacing one of the columns in B by a column* $\mathbf{a}_j$ *(for which* $x_{ij} > 0$*) in A but not in B. Then if* $z_j - c_j < 0$, *the new value of the objective function (denoted by* $\hat{z}$*) will be greater than z.*

Proof. The value of the objective function for the original basic feasible solution is

$$z = \mathbf{C}_B\mathbf{X}_B = (c_{B1}, c_{B2}, ..., c_{Bm})\,(x_{B1}, x_{B2}, ..., x_{Bm}) = \sum_{i=1}^{m} c_{Bi}\, x_{Bi}. \qquad \text{...(A-33)}$$

The new value is

$$\hat{z} = \hat{\mathbf{C}}_B \hat{\mathbf{X}}_B = \sum_{i=1}^{m} \hat{c}_{Bi}\,\hat{x}_{Bi} = \sum_{i=1,\, i\neq r}^{m} \hat{c}_{Bi}\,\hat{x}_{Bi} + \hat{c}_{Br}\,\hat{x}_{Br}$$

where $\hat{c}_{Bi} = c_{Bi}\ (i \neq r)$ and $\hat{c}_{Br} = c_j\ (i = r)$.

Therefore, $$\hat{z} = \sum_{i=1,\, i\neq r}^{m} c_{Bi}\,\hat{x}_{Bi} + c_j\,\hat{x}_{Br}.$$

Substituting the values of new variables $\hat{x}_{Bi}$ and $\hat{x}_{Br}$ from (A-26) and (A-27) into the last expression, we get

$$\hat{z} = \sum_{i=1,\, i\neq r}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}}\right) + c_j\frac{x_{Br}}{x_{rj}} \qquad \text{...(A-34)}$$

Since the term for which $i = r$ is

$$c_{Br}\left(x_{Br} - x_{Br}\frac{x_{rj}}{x_{rj}}\right) = 0,$$

we can include it in the summation (A-34) without changing the value of $\hat{z}$, so that

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}}\right) + c_j\frac{x_{Br}}{x_{rj}} = \sum_{i=1}^{m} c_{Bi}x_{Bi} - \frac{x_{Br}}{x_{rj}}\sum_{i=1}^{m} c_{Bi}x_{ij} + \frac{x_{Br}}{x_{rj}}c_j$$

$$= z - \frac{x_{Br}}{x_{rj}}z_j + \frac{x_{Br}}{x_{rj}}c_j = z - (z_j - c_j)\frac{x_{Br}}{x_{rj}}$$

$$= z - (z_j - c_j)\,v, \text{ where } v = x_{Br}/x_{rj}. \qquad \text{...(A-35)}$$

Now, from (A-35) it is seen that the new value $\hat{z}$ of the objective function is the original value z minus the quantity $(z_j - c_j)\,v$. If $\hat{z}$ is to exceed z then the quantity $z_j - c_j$ must be less than zero, that is, if $z_j - c_j < 0$, the value of the objective function is improved, and thus the theorem is proved.

A.2-1 Basic feasible solution increasing z to maximum possible

So far a procedure has been developed to get a new basic feasible solution which simply improves the value of the objective function z. Arbitrarily selected $\mathbf{a}_j$ (not in B) is to be replaced by β_r in order to get the improved basic feasible solution. At every step of simplex method, main aim is to get such improved basic feasible solution which gives the greatest increase in z. This can be done by selecting among all $\mathbf{a}_j$ (not in B) such vector $\mathbf{a}_k$ to be replaced by β_r which will increase $\mathbf{z}$ maximum possible. So a criterion for selecting the vector $\mathbf{a}_k$ is given in *Theorem (A-5) below.*

Theorem (A-5). *If the vector $\mathbf{a}_k$ to be replaced by β_r, the suffix k can be pre-decided by means of*

$$z_k - c_k = \min\ [\mathbf{z}_j - c_j],\ z_j - c_j < 0.$$

Then the value of z is increased as much as possible for the new basic feasible solution,

Proof. In *Theorem* (A-4), the improved value of z is obtained as

$$\hat{z} = z - \frac{x_{Br}}{x_{rj}}(z_j - c_j)$$

Thus in order to give maximum value of $\hat{z}$, select that value of j (say k) for which the term $(x_{Br}/x_{rj})\ (z_j - c_j)$ is minimum.

The computational difficulty arises while obtaining min $\frac{x_{Br}}{x_{rj}}(z_j - c_j)$, because it is necessary to compute x_{Br}/x_{rj} for each a_j having $z_j - c_j < 0$ by the formula :

$$\frac{x_{Br}}{x_{rj}} = \min_{i}\left(\frac{x_{Bi}}{x_{ij}},\ x_{ij} > 0\right)$$

The change in objective function depends on x_{Br}/x_{rj} and $z_j - c_j$.

Thus to avoid large number of computations of x_{Br}/x_{rj} we can neglect the consideration of x_{Br}/x_{rj}.

Hence the most convenient and time saving device for choosing the vector $\mathbf{a}_k$ to enter the basis B consists of selecting the smallest $z_j - c_j$. This is equivalent to choosing the vector $\mathbf{a}_k$ to replace β_r by means of

$$z_k - \mathbf{c}_k = \min_{j}\ [\mathbf{z}_j - c_j],\ z_j - c_j < 0. \qquad \text{...(A-36)}$$

Thus the theorem is proved.

Note. The following are the advantages of using the criterion (A-36) :

1. The choice of vector $\mathbf{a}_k$ to enter the basis B by using (A-36) gives the greatcst possible increase in z at each step.
2. Once a vector $\mathbf{a}_k$ had been inserted into B, it would never have to be removed again, although no criteria has been developed to guarantee this fact.
3. More than m iterations will not be needed to reach the optimal basic feasible solution.
4. It saves considerable time by giving the required solution in least number of steps.

A.2-2. Implications of degenerate original basic feasible solution

Up to this stage the original basic feasible solution was assumed non-degenerate. Now it remains to examine the implications of *degenerate* original basic feasible solutions

Theorem (A-6). *Given a degenerate basic feasible solution* $\mathbf{X}_B = B^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$. *If for any column* $\mathbf{a}_j$ *in A but not in B, at least one* $x_{ij} > 0$ $(i = 1, 2, ..., m)$, *then a new basic feasible solution is obtained which may or may not be degenerate, by replacing one of the columns in* **B** *by* $\mathbf{a}_j$.

Proof. Follow the proof the *Theorem* (A-3) up to equation (A-31). If some of the $x_{ij} > 0$ correspond to $x_{Bi} = 0$, then $v = 0 = x_{Br}$ and a basic feasible solution will exist and it will be degenerate. Since $\mathbf{x}_{Br} = 0$, (A-26) and (A-27) show that the values of the variables which have been retained in the new solution are unchanged ($\hat{x} = x_{Bi},\ i \neq r$). On the other hand, if all $x_{ij} > 0$ correspond to $x_{Bi} > 0$ none of the $x_{Bi} = 0$ enter into (A-31), so that $v > 0$ and , from [(A-26) and (A-27)], the new solution will be non-degenerate, provided that

$$\min_{i}\left(\frac{x_{Bi}}{x_{ij}},\ x_{ij} > 0\right) \text{ is unique.}$$

Theorem (A-7). *Suppose a degenerate basic feasible solution* $\mathbf{X}_B = B^{-1}\mathbf{b}$ to $A\mathbf{X} = \mathbf{b}$. *If from any column* a_j *in A but not in B, at least one* $x_{ij} \neq 0$ $(i = 1, 2, ..., m)$ *and if the corresponding* $x_{Bi} = 0$, *then a new degenerate basic feasible solution can be obtained.*

Proof. Follow the proof of the *Theorem* (A-3) up to equations (A-28, A-29). If $x_{Br} = 0$, then as long as $x_{rj} \neq 0$, equations (A-28 , A-29) are satisfied, and $\mathbf{a}_j$ can be substituted for β_r to yield a new degenerate (since $x_{Br}/x_{rj} = 0$) basic feasible solution in which $\hat{x}_{Bi} = x_{Bi}$ $(i \neq r)$.

Theorem (A-8). *Given a degenerate basic feasible solution* $\mathbf{X}_B = B^{-1}\mathbf{b}$ *to* $A\mathbf{X} = \mathbf{b}$ *which gives a value for the objective function of* $z = \mathbf{C}_B\mathbf{X}_B$. *Also, a new basic feasible solution is obtained with* $\mathbf{X}_B = B^{-1}\mathbf{b}$ to $A\mathbf{X} = \mathbf{b}$ *by replacing one of the columns in B by a column* $\mathbf{a}_j$ *(for which some* $x_{ij} > 0$*) in A but not B. Then, if* $z_j - c_j \leq 0$, *the new value of the objective function will be such that* $\hat{z} \geq z$.

Proof. Follow the proof of the *Theorem* (A-4) up to equation (A-25). From *Theorem* (A-6) we know that $(z_j - c_j) \leq 0$ (since $v > 0$ and $z_j - c_j \leq 0$), implying that $\hat{z} \geq z$.

A.2-3 Summary of theorems (A-7) to (A-8)

To sum up this section, it is proved so far that given a basic (degenerate or non-degenerate) feasible solution $\mathbf{X}_B = B^{-1}\mathbf{b}$ to $A\mathbf{X} = \mathbf{b}$ which gives a value for the objective function $z = \mathbf{C}_B\mathbf{X}_B$, if for any column $\mathbf{a}_j$ in A but not in B, $z_j - c_j < 0$ and if at least one $x_{ij} > 0$ $(i = 1, 2, ..., m)$ then a new basic feasible solution can be obtained by replacing one of the columns in B by $\mathbf{a}_j$ and the new value $(\hat{z})$ of the objective function will be such that $\hat{z} \geq z$.

Furthermore, to bring the maximum increase in z, we should select a particular vector $\mathbf{a}_k$ from all $\mathbf{a}_j$ (in A but not in B) to enter the basis matrix B for which the suffix k is pre-determined by using the formula

$$z_k - c_k = \min_j (z_j - c_j).$$

A-3. CONDITIONS FOR LP PROBLEM TO POSSESS UNBOUNDED SOLUTION

In theorems given earlier, it has been proved that for $\mathbf{a}_j$ inserted not in the basis matrix B there is at least one $x_{ij} > 0$, $i = 1, 2, ..., m$.

Now, the important point is : what will be the implication if for at least one $\mathbf{a}_j$ all $x_{ij} \leq 0$. This has been proved in the following theorem.

Theorem (A-9). (Unbounded Solution). *Given any feasible solution* $\mathbf{X}_B = B^{-1}\mathbf{b}$ to $A\mathbf{X} = \mathbf{b}$. *If for this solution there is some column* $\mathbf{a}_j$ *in A but not in B for which* $z_j - c_j < 0$ *and* $x_{ij} \leq 0$ $(i = 1, 2, .., m)$ *then if, the objective function is to be maximized, the problem has an unbounded solution.*

Proof. Insert $\mathbf{a}_j$ in B. It is given that the vector $\mathbf{a}_j$ has all $x_{ij} \leq 0$. Since $\mathbf{X}_B = B^{-1}\mathbf{b}$ is the basic feasible solution to $A\mathbf{X} = \mathbf{b}$, therefore from equation (A-23)

$$\sum_{i=1}^{m} x_{Bi}\,\beta_i = \mathbf{b} \qquad \text{...(A-37)}$$

The value of the objective function is

$$z = \mathbf{C}_B\mathbf{X}_B = \sum_{i=1}^{m} c_{Bi}x_{Bi}$$

Let λ be any scalar. If $\lambda\mathbf{a}_j$ be added and subtracted in (A-37), then we get

$$\sum_{i=1}^{m} x_{Bi}\,\beta_i - \lambda\mathbf{a}_j + \lambda\mathbf{a}_j = \mathbf{b} \qquad \text{...(A-39)}$$

Since, $$\mathbf{a}_j = \sum_{i=1}^{m} x_{ij}\,\beta_i, \quad \text{(from A–20)}$$

we have $$-\lambda\mathbf{a}_j = -\lambda\sum_{i=1}^{m} x_{ij}\,\beta_i \qquad \text{...(A-40)}$$

Now substituting the value of $-\lambda\mathbf{a}_j$ from (A-40) in (A-39), we get

$$\sum_{i=1}^{m} x_{Bi}\,\beta_i - \lambda\sum_{i=1}^{m} x_{ij}\,\beta_i + \lambda\mathbf{a}_j = \mathbf{b} \quad \text{or} \quad \sum_{i=1}^{m} (x_{Bi} - \lambda x_{ij})\,\beta_i + \lambda\mathbf{a}_j = \mathbf{b} \qquad \text{...(A.41)}$$

Thus, (A-31) gives the new solution whose variables are

$$\left.\begin{aligned}\hat{x}_{Bi} &= x_{Bi} - \lambda x_{ij},\ i = 1, 2, \ldots, m\\ \hat{x}_{m+1} &= \lambda\end{aligned}\right\} \qquad \ldots\text{(A-42)}$$

Since all $x_{ij} \leq 0$, $x_{Bi} - \lambda x_{ij} \geq 0$ for $\lambda > 0$, so that (A-42) is a feasible solution in which the number of positive variables are less than or equal to $m + 1$ (less than $m + 1$ because some $x_{Bi} - \lambda x_{ij}$ may be zero). In case, the number of positive variables in (A-42) is equal to $m + 1$ (which is greater than m, the number of constraint equations), the solution of (A-42) will be non-basic feasible solution.

Now, the new value (denoted by $\hat{z}$) of the objective function corresponding to new solution (A-42) becomes

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}\,(x_{Bi} - \lambda x_{ij}) + c_j\lambda = \sum_{i=1}^{m} c_{Bi}\,x_{Bi} - \lambda\left\{\sum_{i=1}^{m} c_{Bi}\,x_{ij} - c_j\right\}$$

or

$$\hat{z} = z - \lambda(z_j - c_j) \qquad \ldots\text{(A-43)}$$

Since it is given that $z_j - c_j < 0$, the value of $\hat{z}$ [given by (A-43)] can be made as large as we please by giving λ a sufficiently large value. But, by definition, a linear programming problem has an ***unbounded solution*** if the value of objective function can be made arbitrarily large, so the problem has no finite maximum value of z. Hence the theorem is proved.

EXAMINATION PROBLEM

1. When is a linear programming problem called unbounded ? When A be an $m \times n$ real matrix of rank m. Obtain conditions for the problem : Max $z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq 0$ to be unbounded. **[Meerut (LP) 97 P]**
2. Obtain thye conditions for unboundedness of the LP problem : Max. $z = \mathbf{C}^T\mathbf{X}$, subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq 0$ where $A \in R^{m \times n}$, $x \in R^n$, $b \in R^n$, $c \in R^n$. **[Meerut (LP) 96 BP]**

A-4. CONDITION WHEN IMPROVED BASIC FEASIBLE SOLUTION BECOMES OPTIMAL

In the absence of degeneracy, we have proved [in *Theorem (A-3)* and *(A-4)*] that if for any column $\mathbf{a}_j$ in A but not in B we have $z_j - c_j < 0$, and if at least one $x_{ij} > 0$ $(i = 1, 2, \ldots, m)$, then we can improve the current basic feasible solution such that the new value of the objective function will be such that $\hat{z} > z$.

But this process of improving a given basic feasible solution cannot be continued indefinitely because, there exist only a finite number of basic feasible solutions. Also, in the absence of degeneracy no basis matrix can ever be repeated because the new value of z increases at each round of improvement, and the same basis matrix cannot give two different values of z.

From above, we conclude that the process of improvement in z cannot be continued further as soon as we reach one of the following two possibilities :

(*i*) $z_j - c_j < 0$. for at least one j for which $\mathbf{a}_j$ is not in the basis matrix B and corresponding to this j, all $x_{ij} \leq 0$ $(i = 1, 2, \ldots, m)$

(*ii*) $z_j - c_j \geq 0$, for all $\mathbf{a}_j$ in A but not in $\boldsymbol{B}$.

During the process of improving the value of z, if we reach the condition (*i*) at any step we get an unbounded solution (as already proved in *Theorem* (A-29).

Now we hope that we will get an optimal solution if our process of improving the value of z terminates with condition (*ii*). For this, a theorem for optimality condition has been dealt with :

Theorem (A-10). (Optimality Conditions). *Suppose a basic (degenerate or non-degenerate) feasible solution* $\mathbf{X}_B = \boldsymbol{B}^{-1}\,\mathbf{b}$ to $\mathbf{AX} = \mathbf{b}$ *with* $z^* = \mathbf{C}_B\,\mathbf{X}_B$ *is obtained at any iteration of simplex method. If* $z_j - c_j \geq 0$ *for every column* $\mathbf{a}_j$ *in A but not in* $\mathbf{B}$, *then* $\mathbf{z}^*$ *is the optimum value of the objective function* $z = \mathbf{CX}$ and $\mathbf{X}_B$ *is an optimal basic feasible solution.*

Proof. Given that at any iteration of simplex method, the basic feasible solution is

$$\mathbf{X}_B = (x_{B1}, x_{B2}, \ldots, x_{Bm}) = \boldsymbol{B}^{-1}\,\mathbf{b}$$

The basis matrix is $\boldsymbol{B} = (\beta_1, \beta_2, .., \beta_m)$ and the value of the objective function for this solution is

$$z^* = \mathbf{c}_B\,\mathbf{x}_B = \sum_{i=1}^{m} c_{Bi}\,x_{Bi} \qquad \ldots\text{(A-44)}$$

Also, at this stage, we are faced with the given situation $z_j - c_j \geq 0$ for those j for which $\mathbf{a}_j$ is not in $\boldsymbol{B}$. Let the value of the objective function be

$$z = \mathbf{CX} = \sum_{j=1}^{n+m} c_j x_j \qquad \text{...(A-45)}$$

for any feasible solution $\mathbf{X} = (x_1, x_2, ..., x_{n+m})$ of $A\mathbf{X} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$.

Now in order to prove $z^* \geq z$ in this theorem, following steps are necessary :

$$\mathbf{X}_B = \boldsymbol{B}^{-1}\mathbf{b} = \boldsymbol{B}^{-1}(A\mathbf{X}) = (\boldsymbol{B}^{-1}A)\,\mathbf{X} \quad \text{(because } A\mathbf{x} = \mathbf{b})$$

$$= \boldsymbol{B}^{-1}(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_{n+m}) \bullet \mathbf{X}$$

$$= (B^{-1}\mathbf{a}_1, B^{-1}\mathbf{a}_2, ..., B^{-1}\mathbf{a}_{n+m}) \bullet (x_1, x_2, ..., x_{n+m})$$

$$= (\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_{n+m})\,(x_1, x_2, ..., x_{n+m})$$

$$= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{1,m+n} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2,m+n} \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mj} & \cdots & x_{m,m+n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_{n+m} \end{bmatrix}$$

or $\quad (x_{B1}, ..., x_{Bi}, ..., x_{Bm}) = \left(\sum_{j=1}^{m+n} x_{1j}x_j, \sum_{j=1}^{m+n} x_{2j}x_j, ..., \sum_{j=1}^{m+n} x_{ij}x_j, ..., \sum_{j=1}^{m+n} x_{mj}x_j,\right)$ (multiplying the matrices on the right hand side)

Now, equating ith component on both sides, we get

$$x_{Bi} = \sum_{j=1}^{n+m} x_{ij}x_j \qquad \text{...(A-46)}$$

It is given that $z_j - c_j \geq 0$ for which $\mathbf{a}_j$ is not in $\boldsymbol{B}$. If it is possible to prove that this inequality also holds for all those j for which $\mathbf{a}_j$ is in $\boldsymbol{B}$, then $z_j - c_j \geq 0$ holds for all j for which $\mathbf{a}_j$ may also be in the basis matrix $\boldsymbol{B}$. To show this, consider $\beta_i = \mathbf{a}_j$ then

$$x_j = \mathbf{B}^{-1}\mathbf{a}_j = \boldsymbol{B}^{-1}\beta_i = e_i \quad (e_i \text{ denotes the unit vector whose } i\text{th component is unity})$$

For such j, $z_j = \mathbf{c}_B\mathbf{x}_j = \mathbf{C}_B e_i = (c_{B1}, c_{B2}, ..., c_{Bi}, ..., c_{Bm})\,(0, 0, ..., ..., 1, 0, ..., 0)$ ith component

$$= c_{Bi} = c_j \qquad \text{(because } \beta_i = \mathbf{a}_j)$$

Thus, $z_j - c_j = 0$ for those j for which $\mathbf{a}_j$ is in the basis B.

Hence $z_j - c_j \geq 0$, for all $j = 1, 2, .., m+n$ or $c_j \leq z_j$.

Now multiplying both sides by $\mathbf{x}_j$, $\quad c_j\mathbf{x}_j \leq z_j\mathbf{x}_j \quad$ (since $\mathbf{x}_j \geq \mathbf{0}$ is feasible solution)

or $$\sum_{j=1}^{m+n} c_j x_j \leq \sum_{j=1}^{m+n} z_j x_j \quad \text{or} \quad z \leq \sum_{j=1}^{m+n} (\mathbf{C}_B\mathbf{X}_j)\,x_j$$

or $$z \leq \sum_{j=1}^{m+n} x_j \left(\sum_{i=1}^{m} c_{Bi}\,x_{ij}\right) \qquad \text{[using (5.6) p. 118]}$$

or $$z \leq \sum_{i=1}^{m} c_{Bi} \left(\sum_{j=1}^{m+n} x_j x_{ij}\right)$$

or $$z \leq \sum_{i=1}^{m} c_{Bi}x_{Bi} \qquad \text{[using (A-46)]}$$

or $$z \leq z^* \qquad \text{[using (A-44)]}$$

which was to be proved. Thus, the theorem is now completely established

Note. One important point to be noted here is that this proof did not repquire x_j to be non-degenerate, so the theorem holds for both degenerate and non-degenerate solutions.

Q. 1. Given basic feasible solution to a linear programming problem, show how we can improve this basic feasible solution. State the conditions for optimality.
[**Hint.** See Theorems (A-3), (A-4) and (A-10)]

2. Given a general linear programming problem, explain how you would test whether a basic feasible solution is an optimal solution or not.

3. If for a basic feasible solution $\mathbf{x} = (\mathbf{x}_B, \mathbf{0})$ for a given problem min. $\mathbf{z} = \mathbf{cx}$, subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ it is true that $z_j - c_j \geq 0$ for all j, then prove that the solution is optimum. **[Agra 93]**

A.5. ALTERNATIVE OPTIMAL SOLUTIONS

Definition. *If the set of variables giving the optimal value of the objective function is not unique, alternative optimum solutions to the given linear programming problem exist.*

Theorem (A -11) (Conditions for alternative Optimum Solutions).

(*i*) *If there is an optimal basic feasible solution to a linear programming problem and, for some* $\mathbf{a}_j$ *not in* $\boldsymbol{B}$, $z_j - c_j = 0$, $x_{ij} < 0$ *for all* $i = 1, 2, \ldots, m$, *then non-basic alternative optimum solution will exist.*

(*ii*) *Secondly, if* $z_j - c_j = 0$ *for some* $\mathbf{a}_j$ *not in B and* $x_{ij} > 0$ *for all least one i, then an alternative basic optimum solution will exist.*

Proof. ***First Part*.** Following the proof of *Theorem* (A-9) down to result (A-43),

$$\hat{z} = z^* - \lambda (z_j - c_j) \qquad \left(\text{since } z^* \leq \sum_{i=1}^{m} c_{Bi} x_{Bi} \text{ for given optimal solution}\right)$$

Since, $z_j - c_j = 0$ for some $\mathbf{a}_j$ not in $\boldsymbol{B}$, therefore $\hat{z} = z^*$.

This shows that the non-basic feasible solution given by

$$\left.\begin{aligned} \hat{x}_{Bi} &= x_{Bi} - \lambda x_{ij},\ i = 1, 2, \ldots, m \\ \hat{x}_{m+1} &= \lambda \end{aligned}\right\}$$

when $x_{ij} < 0$ ($i = 1, 2, .., m$) and $\lambda > 0$, gives the same value of z. Hence this new solution with $m + 1$ number of positive variables gives alternative optimum non-basic feasible solutions for arbitrary value of positive scalar λ.

Second Part. Since in this case $x_{ij} > 0$ for at least one i, one β_r can be replaced by $\mathbf{a}_j$ not in $\boldsymbol{B}$ and obtain another basic feasible solution (as explained in *Theorem* (A-4) which gives the value of the objective function as

$$\hat{z} = z^* - \frac{x_{Br}}{x_{rj}} (z_j - c_j) \qquad \text{[from equation (A-35)]}$$

$$= z^* \qquad (\text{because } z_j - c_j = 0)$$

Since the value of the objective function remains unaltered, an alternative optimum basic feasible solution is obtained.

Remark :

1. In case, $\mathbf{x}_B$ is degenerate solution and $z_j - c_j > 0$ for $\mathbf{a}_j$ not in $\boldsymbol{B}$, $x_{Br} = 0$ and $x_{rj} > 0$ then an alternative optimum basic feasible solution exists which can be proved as above after the problem of degeneracy is discussed,
2. Further, notice that if $\mathbf{x}_B$ is non-degenerate and $z_j - c_j > 0$ for all $\mathbf{a}_j$ not in $\boldsymbol{B}$, then only unique optimal soltuion exists. It is necessary that this unique optimal solution must be basic.

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APPENDIX-B

A New Method For Initial Solution Of Transportation Problem

B-1. MATHEMATICAL FORMULATION OF T.P.

There are several methods for finding the *initial basic feasible solution* of Transportation Problem (T.P.) as discussed in **Chapter 12** of **Unit-2.** But, there is no suitable answer to the question : which method is the beast one ? In this article, we have developed a new technique for finding the nearly optimal solution which requires less iterations to reach optimality in comparison to the methods available in the literature. The degeneracy problem is also avoided by this method. This method is better than *Vogel's Approximation Method* also, and is based on the min-max (max-min) criteria of '*Game Theory*'

Mathematical formulation of Transportation Problem has been given in **Chapter 12** of **Unit-2.**

B-2 MIN(MIN-MAX) ALGORITHM*

Step 1. Choose the maximum cost (c_{ij}) cell ($i = 1, ..., m ; j = 1, ..., n$). In case of ties, choose the maximum arbitrarily.

Step 2. (*i*) Choose the minimum cost cell in the row containing the maximum (c_{ij}).

(*ii*) Allocate the maximum possible quantity to the minimum cost cell.

(*iii*) Find the total cost of this allocation.

Step 3. Repeat *Step* 2 for the column cell containing the maximum (c_{ij}).

Step 4. Choose for allocation the cell with minimum cost. In case of a tie choose arbitrarily.

Step 5. Delete the row (column) when the allocation becomes complete. In case of a tie delete either row or column. If the row and column both deserve to be deleted, put zero in the minimum cost cell of either row of column which are not yet deleted.

Example : Consider the problem :

19	30	50	10	7
70	30	40	60	9
40	8	70	20	18
5	8	7	14	

Proceeding step-by-step according to the algorithm suggested above, we get the following solution :

19 **5**	30	50	10 **2**	7
70	30	40 **7**	60 **2**	9
40	8 **8**	70	20 **10**	18
5	8	7	14	

The initial cost = 779 units.

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*This method was developed by the author in 1989 when he was working on deputation as Professor of Operations Research in the University of Salahaddin, Arbil, Iraq. On this article the author was awarded by the *"Ministry of Higher Education and Scientific Research.*

However, if we find the initial solution of this problem by *Vogel's Method* we reach the same solution. But this new approach requires less computations as well as the total number of basic cells are always $m + n - 1$, thus avoiding the problem of degeneracy.

B-3. MAX (MIN-MAX) ALGORITHM

Step 1. Choose the max (c_{ij}) cell $(i = 1, ..., m ; j = 1, ..., n)$. In case of ties, choose the maximum arbitrarily.

Step 2. (*i*) Choose the minimum cost cellin the now containing the maximum (c_{ij})

(*ii*) Allocate the maximum possible quantity to the minimum cost cell.

(*iii*) Find the total cost of this allocation.

Step 3. Repeat *Step 2* for the column cell containing the maximum (c_{ij}).

Step 4. Choose for allocation the cell with maximum total cost. In case of a tie choose arbitrarily.

Step 5. Delete the row (column) when the allocation becomes complete. In case of tie delete either row or column. If the row and column both deserve do be delete, put zero in the minimum cost cell which are not yet deleted.

Step 6. When all $m + n - 1$ cells are allocated, stop. Otherwise, go to *Step* 1.

B-4. VERIFICATION BY EXAMPLE

Following example verifies that the initial solution becomes optimal when

max (min – max) = min (min – max).

Conisder the following initial BFS obtained by both the above algorithms :

4 **2**	1 **3**	4	2	5
5	2 **6**	5	3	6
6	3	1 **5**	1 **2**	7
2	9	5	2	

Max		**Min**	**Max**
6	Row	$1 \times 5 = 5$	
	Column	$4 \times 2 = 8^*$	a_{11}, Omit column 1
5	Row	$2 \times 6 = 12^*$	a_{22}, Omit Row 2
	Column	$1 \times 5 = 5$	
3	Row	$1 \times 2 = 2$	
	Column	$1 \times 3^*$	a_{12}, Omit Row 1

Note : To avoid degeneracy we allocate empty (zero) to the cell a_{32} and we omit column 2.

1 Row and Column $1 \times 2 = 2$

which is the last cell solution. Total cost is 30 and this solution satisfies the test of optimality.

B-5. CONCLUDING REMARK

The algorithm developed in this article has the following major advantages :

(*i*) It requires less computations in comparison to the methods existing in the literature.

(*ii*) The problem of degeneracy will not arise in the initial BFS.

(*iii*) The algorithm has the ***interesting property*** : max (min – max) = min (min – max) = optimum solution.

***

APPENDIX-C

NUMERICAL TABLES
(Table 1 - Table 12)

Table 1. Logarithms

Table 2. Antilogarithms

Table 3. Power, Roots and Reciprocal

Table 4. Negative Exponential Function

Table 5. Interest Tables (CAF, PWF, CAFS, SFF, CRF, PWFS)

Table 6. Random Number Tables

Table 7. Random Normal Numbers

Table 8. Areas under the Normal curve

Table 9. Proportion of Total Area under the Normal curve from ∞ to t, where $t = (x - \mu)/\sigma$

Table 10. Values of e^x and e^{-x}

Table 11. Poisson Distribution

Table 12. Table of Control Charts

Appendix Table 1–A : Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Appendix Table 1-B : Logarithms (Contd.)

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	2	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9788	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Appendix Table 2-A : Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2696	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Appendix Table 2-B : Antilogarithms (Contd.)

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4256	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	7237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6497	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Appendix Table 3-A

Reciprocals of Numbers : From 1 To 4.49

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
1.0	1.000	9901	9804	9709	9615	9524	9434	9346	9259	9174									
1.1	.9091	9009	8929	8850	8772	8696	8621	8547	8475	8403									
1.2	.8333	8264	8197	8130	8065	8000	7937	7874	7813	7752									
1.3	.7692	7634	7576	7519	7463	7407	7353	7299	7246	7194									
1.4	.7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
1.5	.6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
1.6	.6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
1.7	.5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	6	10	13	16	20	23	26	29
1.8	.5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1.9	.5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
2.0	.5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
2.1	.4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	20
2.2	.4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2.3	.4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2.4	.4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
2.5	.4000	3984	3968	3953	3937	3922	3906	3891	3896	3861	2	3	5	6	8	9	11	12	14
2.6	.3846	3831	3817	3802	3788	3774	3759	3745	3731	3718	1	3	4	6	7	8	10	11	13
2.7	.3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
2.8	.3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
2.9	.2448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
3.0	.3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	7	9	10
3.1	.3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3.2	.3125	3115	3106	3090	3086	3977	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3.3	.3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
3.4	.2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
3.5	.2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	2	3	4	5	6	6	7
3.6	.2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
3.7	.2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
3.8	.2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
3.9	.2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
4.0	.2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	6
4.1	.2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
4.2	.2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
4.3	.2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
4.4	.2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
4.5	.2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
4.6	.2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
4.7	.2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
4.8	.2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
4.9	.2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
5.0	.2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
5.1	.1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
5.2	.1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
5.3	.1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
5.4	.1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

(Numbers in difference columns to be subtracted, not added.)

Appendix Table 3-B

Reciprocals of Numbers : From 5.5 to 9.99 (Contd.)

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
5.5	.1818	1815	1812	1808	1805	1802	1799	1795	1792	1789	0	1	1	1	2	2	2	3	3
5.6	.1786	1783	1779	1776	1773	1770	1767	1764	1761	1757	0	1	1	1	2	2	2	3	3
5.7	.1754	1751	1748	1745	1742	1739	1736	1733	1730	1727	0	1	1	1	1	2	2	2	3
5.8	.1724	1721	1718	1715	1712	1709	1706	1704	1701	1698	0	1	1	1	1	2	2	2	3
5.9	.1695	1692	1689	1686	1684	1681	1678	1675	1672	1669	0	1	1	1	1	2	2	2	3
6.0	.1667	1664	1661	1658	1656	1653	1650	1647	1645	1642	0	1	1	1	1	2	2	2	3
6.1	.1639	1637	1634	1631	1629	1626	1623	1621	1618	1616	0	1	1	1	1	2	2	2	2
6.2	.1613	1610	1608	1605	1603	1600	1597	1595	1592	1590	0	1	1	1	1	2	2	2	2
6.3	.1587	1585	1582	1580	1577	1575	1572	1570	1567	1565	0	0	1	1	1	1	2	2	2
6.4	.1562	1560	1558	1555	1553	1550	1548	1546	1543	1541	0	0	1	1	1	1	2	2	2
6.5	.1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0	0	1	1	1	1	2	2	2
6.6	.1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0	0	1	1	1	1	2	2	2
6.7	.1493	1490	1488	1486	1484	1481	1479	1477	1475	1373	0	0	1	1	1	1	2	2	2
6.8	.1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0	0	1	1	1	1	2	2	2
6.9	.1449	1447	1445	1443	1441	1439	1437	1435	1433	1431	0	0	1	1	1	1	2	2	2
7.0	.1429	1427	1425	1422	1420	1418	1416	1414	1412	1410	0	0	1	1	1	1	1	2	2
7.1	.1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0	0	1	1	1	1	1	2	2
7.2	.1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0	0	1	1	1	1	1	2	2
7.3	.1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0	0	1	1	1	1	1	2	2
7.4	.1351	1350	1348	1346	1344	1342	1340	1339	1337	1335	0	0	1	1	1	1	1	1	2
7.5	.1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0	0	1	1	1	1	1	1	2
7.6	.1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0	0	1	1	1	1	1	1	2
7.7	.1299	1297	1295	1294	1292	1290	1289	1287	1285	1284	0	0	0	1	1	1	1	1	1
7.8	.1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0	0	0	1	1	1	1	1	1
7.9	.1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0	0	0	1	1	1	1	1	1
8.0	.1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0	0	0	1	1	1	1	1	1
8.1	.1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0	0	0	1	1	1	1	1	1
8.2	.1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0	0	0	1	1	1	1	1	1
8.3	.1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0	0	0	1	1	1	1	1	1
8.4	.1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0	0	0	1	1	1	1	1	1
8.5	.1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0	0	0	1	1	1	1	1	1
8.6	.1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0	0	0	1	1	1	1	1	1
8.7	.1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0	0	0	1	1	1	1	1	1
8.8	.1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0	0	0	1	1	1	1	1	1
8.9	.1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0	0	0	1	1	1	1	1	1
9.0	.1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0	0	0	1	1	1	1	1	1
9.1	.1099	1098	1096	1095	1094	1093	1092	1090	1089	1888	0	0	0	0	1	1	1	1	1
9.2	.1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0	0	0	0	1	1	1	1	1
9.3	.1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0	0	0	0	1	1	1	1	1
9.4	.1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0	0	0	0	1	1	1	1	1
9.5	.1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0	0	0	0	1	1	1	1	1
9.6	.1042	1041	1039	1038	1037	1036	1035	1034	1033	1032	0	0	0	0	1	1	1	1	1
9.7	.1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0	0	0	0	1	1	1	1	1
9.8	.1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0	0	0	0	1	1	1	1	1
9.9	.1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0	0	0	0	0	1	1	1	1

(Numbers in difference columns to be subtracted, not added.)

Appendix Table 3-C

Squares. Cubes. Square Roots and Cube Roots

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
					50	2500	125000	7.0711	3.6840
1	1	1	1.0000	1.0000	51	2601	132651	7.1414	3.7084
2	4	8	1.4142	1.2599	52	2704	140608	7.2111	3.7325
3	9	27	1.7321	1.4422	53	2809	148877	7.2801	3.7563
4	16	64	2.0000	1.5874	54	2916	157464	7.3485	3.7798
5	24	125	2.2361	1.7100	55	3025	166375	7.4162	3.8030
6	36	216	2.4495	1.8171	56	3136	175616	7.4833	3.8259
7	49	343	2.6458	1.9129	57	3249	185195	7.5498	3.8485
8	64	512	2.8284	2.0000	58	3364	195112	7.6158	3.8709
9	81	729	3.0000	2.0801	59	3481	205379	7.6811	3.8930
10	100	1000	3.1632	2.1544	60	3600	216000	7.7460	3.9149
11	121	1331	3.3166	2.2240	61.	3721	226981	7.8102	3.9365
12	144	1728	3.4641	2.2894	62	3844	238328	7.8740	3.9579
13	169	2197	3.6056	2.3513	63	3969	250047	7.9373	3.9791
14	196	2744	3.7417	2.4101	64	4069	262144	8.0000	4.0000
15	225	3375	3.7830	2.4662	65	4225	274625	8.0623	4.0207
16	256	4096	4.0000	2.5198	66	4356	287496	8.1240	4.0412
17	289	4913	4.1231	2.5713	67	4489	300763	8.1854	4.0615
18	324	5832	4.2426	2.6207	68	4624	314432	8.2462	4.0817
19	361	6859	4.3589	2.6684	69	4761	328509	8.3066	4.1016
20	400	8000	4.4721	2.7144	70	4900	343000	8.3666	4.1213
21	441	9261	4.5826	2.7589	71	5041	357911	8.4261	4.1408
22	484	10648	4.6904	2.8020	72	5184	373248	8.4853	4.1602
23	529	12167	4.7958	2.8439	73	5329	389017	8.5440	4.1793
24	576	13824	4.8990	2.8845	74	5476	405224	8.6023	4.1983
25	625	15625	5.0000	2.9240	75	5625	421875	8.6603	4.2172
26	676	17576	5.0990	2.9625	76	5776	438976	8.7178	4.2358
27	729	19683	5.1962	3.0000	77	5929	456533	8.7750	4.2543
28	784	21952	5.2915	3.0366	78	6084	474552	8.8318	4.2727
29	841	24389	5.3852	3.0723	79	6241	493039	8.8882	4.2908
30	900	27000	5.4772	3.1072	80	6400	512000	8.9443	4.3089
31	961	29791	5.5678	3.1414	81	6561	531441	9.0000	4.3267
32	1024	32768	5.6569	3.1748	82	6724	551368	9.0554	4.3445
33	1089	35937	5.7446	3.2075	83	6889	571787	9.1104	4.3621
34	1156	39304	5.8310	3.2396	84	7056	592704	9.1652	4.3795
35	1225	42875	5.9161	3.2711	85	7225	614125	9.2195	4.3968
36	1296	46656	6.0000	3.3019	86	7396	636956	9.2736	4.4140
37	1369	50653	6.0828	3.3322	87	7569	658503	9.3274	4.4310
38	1444	54872	6.1644	3.3620	88	7744	681472	9.3808	4.4480
39	1521	59319	6.2450	3.3912	89	7921	704969	9.4340	4.4647
40	1600	64000	6.3246	3.4200	90	8100	729000	9.4868	4.4814
41	1681	68921	6.4031	3.4482	91	8281	753571	9.5394	4.4979
42	1764	74088	6.4807	3.4760	92	8464	778688	9.5917	4.5144
43	1849	79507	6.5574	3.5034	93	8649	804357	9.6437	4.5307
44	1936	85184	6.6332	3.5303	94	8836	830584	9.6954	5.5468
45	2025	91125	6.7082	3.5569	95	9025	857375	9.7468	4.5629
46	2116	97336	6.7823	3.5830	96	9216	884736	9.7980	4.5789
47	2209	103823	6.8557	3.6088	97	9409	912673	9.8489	4.5949
48	2304	110592	6.9282	3.6342	98	9604	941192	9.8995	4.6104
49	2401	117649	7.0000	3.6593	99	9801	970299	9.9499	4.6261

Appendix Table 4-A

Negative Exponential Function : exp $(-x)$ 0.000 to 0.349

	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.00	1.0000	0.9990	0.9980	0.9970	0.9960	0.9950	0.9940	0.9930	0.9920	0.9910
0.01	0.9900	0.9891	0.9881	0.9871	0.9861	0.9851	0.9841	0.9831	0.9822	0.9812
0.02	0.9802	0.9792	0.9782	0.9773	0.9763	0.9753	0.9743	0.9734	0.9724	0.9714
0.03	0.9704	0.9695	0.9685	0.9675	0.9666	0.9656	0.9646	0.9637	0.9627	0.9618
0.04	0.9608	0.9598	0.9589	0.9579	0.9570	0.9560	0.9550	0.9541	0.9531	0.9522
0.05	0.9512	0.9503	0.9493	0.9484	0.9474	0.9465	0.9455	0.9446	0.9436	0.9427
0.06	0.9418	0.9408	0.9399	0.9389	0.9380	0.9371	0.9361	0.9352	0.9343	0.9333
0.07	0.9324	0.9315	0.9305	0.9296	0.9287	0.9277	0.9268	0.9259	0.9250	0.9240
0.08	0.9231	0.9222	0.9213	0.9204	0.9194	0.9185	0.9176	0.9167	0.9158	0.9148
0.09	0.9139	0.9130	0.9121	0.9112	0.9103	0.9094	0.9085	0.9076	0.9066	0.9057
0.10	0.9048	0.9039	0.9030	0.9021	0.9012	0.9003	0.8994	0.8985	0.8976	0.8967
0.11	0.8958	0.8949	0.8940	0.8932	0.8923	0.8914	0.8905	0.8896	0.8887	0.8878
0.12	0.8869	0.8860	0.8851	0.8843	0.8834	0.8825	0.8816	0.8807	0.8799	0.8790
0.13	0.8781	0.8772	0.8763	0.8755	0.8746	0.8737	0.8728	0.8720	0.8711	0.8702
0.14	0.8694	0.8685	0.8676	0.8668	0.8659	0.8650	0.8642	0.8633	0.8624	0.8616
0.15	0.8607	0.8598	0.8590	0.8581	0.8573	0.8564	0.8556	0.8547	0.8538	0.8530
0.16	0.8521	0.8513	0.8504	0.8496	0.8487	0.8479	0.8470	0.8462	0.8454	0.8445
0.17	0.8437	0.8428	0.8420	0.8411	0.8403	0.8395	0.8386	0.8378	0.8369	0.8361
0.18	0.8353	0.8344	0.8336	0.8328	0.8319	0.8311	0.8303	0.8294	0.8286	0.8278
0.19	0.8270	0.8261	0.8253	0.8245	0.8237	0.8228	0.8220	0.8212	0.8204	0.8195
0.20	0.8187	0.8179	0.8171	0.8163	0.8155	0.8146	0.8138	0.8130	0.8122	0.8114
0.21	0.8106	0.8098	0.8090	0.8082	0.8073	0.8065	0.8057	0.8049	0.8041	0.8033
0.22	0.8025	0.8017	0.8009	0.8001	0.7993	0.7985	07977	0.7969	0.7961	0.7953
0.23	0.7945	0.7937	0.7929	0.7922	0.7914	0.7906	0.7898	0.7890	0.7882	0.7874
0.24	0.7866	0.7858	0.7851	0.7843	0.7835	0.7827	0.7819	0.7811	0.7804	0.7796
0.25	0.7788	0.7780	0.7772	0.7765	0.7757	0.7749	0.7741	0.7734	0.7726	0.7718
0.26	0.7711	0.7703	0.7695	0.7687	0.7680	0.7672	0.7664	0.7657	0.7649	0.7641
0.27	0.7634	0.7626	0.7619	0.7611	0.7600	0.7596	0.7588	0.7581	0.7573	0.7565
0.28	0.7558	0.7550	0.7543	0.7535	0.7528	0.7520	0.7513	0.7505	0.7498	0.7490
0.29	0.7483	0.7475	0.7468	0.7460	0.7453	0.7445	0.7438	0.7430	0.7423	0.7416
0.30	0.7408	0.7401	0.7393	0.7386	0.7393	0.7371	0.7364	0.7357	0.7349	0.7342
0.31	0.7334	0.7327	0.7320	0.7312	0.7305	0.7298	0.7291	0.7283	0.7276	0.7269
0.32	0.7261	0.7254	0.7247	0.7240	0.7233	0.7225	0.7218	0.7211	0.7204	0.7196
0.33	0.7189	0.7182	0.7175	0.7168	0.7161	0.7153	0.7146	0.7139	0.7132	0.7125
0.34	0.7118	0.7111	0.7103	0.7096	0.7089	0.7082	0.7075	0.7068	0.7061	0.7054

Appendix Table 4-B

Negative Exponential Function : (Continued) 0.350 to 0.699

	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.35	0.7047	0.7040	0.7033	0.7026	0.7019	0.7012	0.7005	0.6998	0.6991	0.6984
0.36	0.6977	0.6970	0.6963	0.6956	0.6949	0.6942	0.6935	0.6928	0.6921	0.6914
0.37	0.6907	0.6900	0.6894	0.6887	0.6880	0.6873	0.6866	0.6859	0.6852	0.6845
0.38	0.6839	0.6832	0.6825	0.6818	0.6811	0.6805	0.6798	0.6791	0.6781	0.6777
0.39	0.6771	0.6764	0.6757	0.6750	0.6744	0.6737	0.6730	0.6723	0.6717	0.6710
0.40	0.6703	0.6697	0.6690	0.6683	0.6676	0.6670	0.6663	0.6656	0.6650	0.6643
0.41	0.6637	0.6630	0.6623	0.6617	0.6610	0.6603	0.6597	0.6590	0.6584	0.6577
0.42	0.6570	0.6564	0.6557	0.6551	0.6544	0.6538	0.6531	0.6525	0.6518	0.6512
0.43	0.6505	0.6499	0.6492	0.6486	0.6479	0.6473	0.6466	0.6460	0.6453	0.6447
0.44	0.6440	0.6434	0.6427	0.6421	0.6415	0.6408	0.6402	0.6395	0.6389	0.6383
0.45	0.6376	0.6370	0.6364	0.6357	0.6351	0.6344	0.6338	0.6332	0.6325	0.6319
0.46	0.6313	0.6307	0.6300	0.6294	0.6288	0.6281	0.6275	0.6269	0.6263	0.6256
0.47	0.6250	0.6244	0.6238	0.6231	0.6225	0.6219	0.6213	0.6206	0.6200	0.6114
0.48	0.6188	0.6182	0.6175	0.6169	0.6163	0.6157	0.6151	0.6145	0.6139	0.6132
0.49	0.6126	0.6120	0.6114	0.6108	0.6102	0.6096	0.6090	0.6084	0.6077	0.6071
0.50	0.6065	0.6059	0.6053	0.6047	0.6041	0.6035	0.6029	0.6023	0.6017	0.6011
0.51	0.6005	0.5999	0.5993	0.5987	0.5981	0.5975	0.5969	0.5963	0.5957	0.5951
0.52	0.5945	0.5939	0.5933	0.5927	0.5921	0.5916	0.5910	0.5904	0.5898	0.5892
0.53	0.5886	0.5880	0.5874	0.5868	0.5863	0.5857	0.5851	0.5845	0.5839	0.5833
0.54	0.5827	0.5822	0.5816	0.5810	0.5804	0.5798	0.5793	0.5787	0.5781	0.5775
0.55	0.5769	0.5764	0.5758	0.5752	0.5746	0.5741	0.5735	0.5729	0.5724	0.5718
0.56	0.5712	0.5706	0.5701	0.5695	0.5689	0.5684	0.5678	0.5672	0.5667	0.5661
0.57	0.5655	0.5650	0.5644	0.5638	0.5633	0.5627	0.5621	0.5616	0.5610	0.2605
0.58	0.5599	0.5593	0.5588	0.5582	0.5577	0.5571	0.5565	0.5560	0.5554	0.5549
0.59	0.5543	0.5538	0.5532	0.5527	0.5521	0.5516	0.5510	0.5505	0.5499	0.5494
0.60	0.5488	0.5483	0.5477	0.5442	0.5466	0.5461	0.5455	0.5450	0.5444	0.5439
0.61	0.5434	0.5428	0.5423	0.5417	0.5412	0.5406	0.5401	0.5396	0.5390	0.5385
0.62	0.5379	0.5374	0.5369	0.5363	0.5358	0.5353	0.5347	0.5342	0.5337	0.5331
0.63	0.5326	0.5321	0.5315	0.5310	0.5305	0.5299	0.5294	0.5289	0.5283	0.5278
0.64	0.5273	0.5268	0.5262	0.5257	0.5252	0.5247	0.5241	0.5236	0.5231	0.5226
0.65	0.5220	0.5215	0.5210	0.5205	0.5200	0.5194	0.5189	0.5184	0.5179	0.5174
0.66	0.5169	0.5163	0.5158	0.5153	0.5148	0.5143	0.5138	0.5132	0.5127	0.5122
0.67	0.5117	0.5112	0.5107	0.5102	0.5097	0.5092	0.5086	0.5081	0.5076	0.5071
0.68	0.5066	0.5061	0.5056	0.5051	0.5046	0.5041	0.5036	0.5031	0.5026	0.5021
0.69	0.5016	0.5011	0.5006	0.5001	0.4996	0.4991	0.4986	0.4981	0.4976	0.4971

Appendix Table 4-C

Negative Exponential Function : (Continued) 0.700 to 0.999

	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.70	0.4066	0.4961	0.4956	0.4951	0.4946	0.4941	0.4936	0.4931	0.4926	0.4921
0.71	0.4916	0.4912	0.4907	0.4902	0.4897	0.4892	0.4887	0.4882	0.4877	0.4872
0.72	0.4868	0.4863	0.4858	0.4853	0.4848	0.4843	0.4838	0.4834	0.4829	0.4824
0.73	0.4819	0.4914	0.4809	0.4805	0.4800	0.4795	0.4790	0.4785	0.4781	0.4776
0.74	0.4771	0.4766	0.4762	0.4757	0.4752	0.4747	0.4743	0.4738	0.4733	0.4728
0.75	0.4724	0.4719	0.4714	0.4710	0.4705	0.4700	0.4695	0.4691	0.4686	0.4681
0.76	0.4677	0.4672	0.4667	0.4663	0.4658	0.4653	0.4649	0.4644	0.4639	0.4635
0.77	0.4630	0.4626	0.4621	0.4616	0.4612	0.4607	0.4602	0.4598	0.4593	0.4589
0.78	0.4584	0.4579	0.4575	0.4570	0.4566	0.4561	0.4557	0.4552	0.4548	0.4543
0.79	0.4538	0.4534	0.4529	0.4525	0.4520	0.4516	0.4511	0.4507	0.4502	0.4498
0.80	0.4493	0.4489	0.4484	0.4480	0.4475	0.4471	0.4466	0.4462	0.4457	0.4453
0.81	0.4449	0.4444	0.4440	0.4435	0.4431	0.4426	0.4422	0.4418	0.4413	0.4409
0.82	0.4404	0.4400	0.4396	0.4391	0.4387	0.4382	0.4378	0.4374	0.4369	0.4365
0.83	0.4360	0.4356	0.4352	0.4347	0.4343	0.4339	0.4334	0.4330	0.4326	0.4321
0.84	0.4317	0.4313	0.4308	0.4304	0.4300	0.4296	0.4291	0.4287	0.4283	0.4278
0.85	0.4274	0.4270	0.4266	0.4261	0.4257	0.4253	0.4239	0.4244	0.4240	0.4236
0.86	0.4232	0.4227	0.4223	0.4219	0.4215	0.4211	0.4206	0.4202	0.4198	0.4194
0.87	0.4190	0.4185	0.4181	0.4117	0.4173	0.4169	0.4164	0.4160	0.4156	0.4152
0.88	0.4148	0.4144	0.4140	0.4135	0.4131	0.4127	0.4123	0.4119	0.4115	0.4111
0.89	0.4107	0.4102	0.4098	0.4094	0.4090	0.4086	0.4082	0.4078	0.4074	0.4070
0.90	0.4066	0.4062	0.4058	0.4054	0.4049	0.4045	0.4041	0.4037	0.4033	0.4029
0.91	0.4025	0.4021	0.4017	0.4013	0.4009	0.4005	0.4001	0.3997	0.3993	0.3989
0.92	0.3985	0.3981	0.3977	0.3973	0.3969	0.3965	0.3961	0.3957	0.3953	0.30949
0.93	0.3946	0.3042	0.3938	0.3934	0.3930	0.3926	0.3922	0.3918	0.3914	0.3910
0.94	0.3906	0.3902	0.3898	0.3895	0.3891	0.3887	0.3883	0.3879	0.3885	0.3871
0.95	0.3867	0.3864	0.3860	0.3856	0.3852	0.3848	0.3844	0.3840	0.3837	0.3833
0.96	0.3829	0.3825	0.3821	0.3817	0.3814	0.3810	0.3806	0.3802	0.3798	0.3795
0.97	0.3791	0.3787	0.3783	0.3779	0.3776	0.3772	0.3768	0.3764	0.3761	0.3761
0.98	0.3753	0.3749	0.3746	0.3742	0.3738	0.3734	0.3731	0.3727	0.3723	0.3719
0.99	0.3716	0.3712	0.3708	0.3705	0.3701	0.3697	0.3694	0.3690	0.3686	0.3682

Appendix Table 4-D

Negative Exponential Function : (Continued) 1.00 to 3.99

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.0	0.36788	0.36422	0.36059	0.35701	0.35345	0.34994	0.34646	0.34301	0.33960	0.33622
1.1	0.33287	0.32956	0.32628	0.32303	0.31982	0.31664	0.31349	0.31037	0.30728	0.30422
1.2	0.30119	0.29820	0.29523	0.29229	0.28938	0.28650	0.20365	0.28083	0.27804	0.27527
1.3	0.27553	0.26982	0.26714	0.26448	0.26185	0.25924	0.25666	0.25411	0.25158	0.24908
1.4	0.24660	0.24414	0.24171	0.23931	0.23693	0.23457	0.23224	0.22993	0.22764	0.22537
1.5	0.22313	0.22091	0.21871	0.21654	0.21438	0.21225	0.21014	0.20805	0.20598	0.20393
1.6	0.20190	0.19989	0.19790	0.19593	0.19398	0.19205	0.19014	0.18825	0.18637	0.18452
1.7	0.18268	0.18087	0.17907	0.17728	0.17552	0.17377	0.17204	0.17033	0.16864	0.16696
1.8	0.16530	0.16365	0.16203	0.16041	0.15882	0.15724	0.15567	0.15412	0.15259	0.15107
1.9	0.14957	0.14808	0.14661	0.14515	0.14370	0.14227	0.14086	0.13946	0.13807	0.13670
2.0	0.13534	0.13399	0.13266	0.13134	0.13003	0.12873	0.12745	0.12619	0.12493	0.12369
2.1	0.12246	0.12124	0.12003	0.11884	0.11765	0.11648	0.11533	0.11418	0.11304	0.11192
2.2	0.11080	0.10970	0.10861	0.10753	0.10646	0.10540	0.10435	0.10331	0.10228	0.10127
2.3	0.10026	0.09826	0.09827	0.09730	0.09633	0.09537	0.09442	0.09348	0.09255	0.09163
2.4	0.09072	0.08982	0.08892	0.08804	0.08716	0.08629	0.08543	0.08458	0.08374	0.08291
2.5	0.08208	0.08127	0.08046	0.07966	0.07887	0.07808	0.07730	0.07654	0.07577	0.07502
2.6	0.07427	0.07353	0.07280	0.07208	0.07136	0.07065	0.06995	0.06925	0.06856	0.06788
2.7	0.06721	0.06654	0.06587	0.06522	0.06457	0.06393	0.06329	0.06266	0.06204	0.06142
2.8	0.06081	0.06020	0.5961	0.05901	0.05843	0.05784	0.05727	0.05670	0.05613	0.05558
2.9	0.05502	0.05448	0.05393	0.05340	0.05287	0.05234	0.05182	0.05130	0.05079	0.05029
3.0	0.04979	0.04929	0.04880	0.04832	0.04783	0.04736	0.04689	0.04642	0.04596	0.04550
3.1	0.04505	0.04460	0.04416	0.04372	0.04328	0.04285	0.04243	0.04200	0.04259	0.04117
3.2	0.04076	0.04036	0.03996	0.03956	0.03916	0.03877	0.03839	0.03801	0.03763	0.03725
3.3	0.03688	0.03652	0.03615	0.03579	0.03544	0.03508	0.03474	0.03439	0.03405	0.03371
3.4	0.03337	0.03304	0.03271	0.03239	0.03206	0.03175	0.03143	0.03112	0.03118	0.03050
3.5	0.03020	0.02990	0.02960	0.02930	0.02901	0.02872	0.02844	0.02816	0.02788	0.02760
3.6	0.02732	0.02705	0.02678	0.02652	0.02625	0.02599	0.02573	0.02548	0.02522	0.02497
3.7	0.02472	0.02448	0.02423	0.02399	0.02375	0.02352	0.02328	0.02305	0.02282	0.02260
3.8	0.02237	0.02215	0.02193	0.02171	0.02149	0.02128	0.02107	0.02086	0.02065	0.02045
3.9	0.02024	0.02004	0.01984	0.01964	0.01945	0.01925	0.01906	0.01887	0.01869	0.01850

Appendix Table 4-E

Negative Exponential Function : (Continued) 4.00 to 6.99

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
4.0	0.01832	0.01813	0.01795	0.01777	0.01760	0.01742	0.01725	0.01708	0.01691	0.01674
4.1	0.01657	0.01641	0.01624	0.01608	0.01692	0.01576	0.01561	0.01545	0.01530	0.01515
4.2	0.01500	0.01485	0.01470	0.01455	0.01441	0.01426	0.01412	0.01398	0.01384	0.01370
4.3	0.01357	0.01343	0.01330	0.01317	0.01304	0.01291	0.01278	0.01265	0.01253	0.01240
4.4	0.01228	0.01216	0.01203	0.01191	0.01180	0.01168	0.01156	0.01145	0.01133	0.01122
4.5	0.01111	0.01100	0.01089	0.01078	0.01067	0.01057	0.01046	0.01036	0,01025	0.01015
4.6	0.01005	0.00955	0.00985	0.00975	0.00966	0.00956	0.00947	0.00937	0.00928	0.00919
4.7	0.00910	0.00900	0.00892	0.00883	0.00874	0.00865	0.00857	0.00848	0.00840	0.00831
4.8	0.00823	0.00815	0.00807	0.00799	0.00791	0.00783	0.00775	0.00767	0.00760	0.00752
4.9	0.00745	0.00737	0.00730	0.00723	0.00715	0.00708	0.00701	0.00694	0.00687	0.00681
5.0	0.00674	0.00667	0.00660	0.00654	0.00647	0.00641	0.00635	0.00628	0.006722	0.00616
5.1	0.00610	0.00604	0.00598	0.00592	0.00586	0.00580	0.00574	0.00568	0.00563	0.00557
5.2	0.00552	0.00546	0.00541	0.00535	0.00530	0.00525	0.00520	0.00514	0.00509	0.00504
5.3	0.00499	0.00494	0.00489	0.00484	0.00480	0.00475	0.00470	0.00465	0.00461	0.00456
5.4	0.00452	0.00447	0.00443	0.00438	0.00434	0.00430	0.00425	0.00421	0.00417	0.00413
5.5	0.00409	0.00405	0.00401	0.00397	0.00393	0.0389	0.00385	0.00381	0.00377	0.00374
5.6	0.00370	0.00366	0.00362	0.00359	0.00355	0.00352	0.00348	0.00345	0.00341	0.00338
5.7	0.00335	0.00331	0.00328	0.00325	0.00321	0.00318	0.0015	0.00312	0.00309	0.00306
5.8	0.00303	0.00300	0.00297	0.00294	0.00291	0.00288	0.00285	0.00282	0.00279	0.00277
5.9	0.00274	0.00271	0.00269	0.00266	0.00263	0.00261	0.00258	0.00255	0.00253	0.00250
6.0	0.00248	0.00245	0.00243	0.00241	0.00238	0.00236	0.00233	0.00231	0.00229	0.00227
6.1	0.00224	0.00222	0.00220	0.00218	0.00215	0.00213	0.00211	0.00209	0.00207	0.00205
6.2	0.00203	0.00201	0.00199	0.00197	0.00195	0.00193	0.00191	0.00189	0.00187	0.00185
6.3	0.00184	0.00182	0.00180	0.00178	0.00176	0.00175	0.00173	0.00171	0.00170	0.00168
6.4	0.00166	0.00165	0.00163	0.00161	0.00160	0.00158	0.00156	0.00155	0.00153	0.00152
6.5	0.00150	0.00149	0.00147	0.00146	0.00144	0.00143	0.00142	0.00140	0.00139	0.00137
6.6	0.00136	0.00135	0.00133	0.00132	0.00131	0.00129	0.00128	0.00127	0.00126	0.00124
6.7	0.00123	0.00122	0.00121	0.00119	0.00118	0.00117	0.00116	0.00115	0.00114	0.00112
6.8	0.00111	0.00110	0.00109	0.00108	0.00107	0.00106	0.00105	0.00104	0.00103	0.00102
6.9	0.00101	0.00100	0.00099	0.00098	0.00097	0.00096	0.00095	0.00094	0.00093	0.00092

Appendix Table 4-F

Negative Exponental Function : (Continued) 7.00 to 9.99

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
7.0	0.00091	0.00090	0.00089	0.00088	0.00088	0.00087	0.00086	0.00085	0.00084	0.00083
7.1	0.00083	0.00082	0.00081	0.00080	0.00079	0.00078	0.00078	0.00077	0.00076	0.00075
7.2	0.00075	0.00074	0.00073	0.00072	0.00072	0.00071	0.00070	0.00070	0.00069	0.00068
7.3	0.00068	0.00067	0.00066	0.00066	0.00065	0.00064	0.00064	0.00063	0.00062	0.00062
7.4	0.00061	0.00061	0.00060	0.00059	0.00059	0.00058	0.00058	0.00057	0.00056	0.00056
7.5	0.00055	0.00055	0.00054	0.00054	0.00053	0.00053	0.00052	0.00052	0.00051	0.00051
7.6	0.00050	0.00050	0.00049	0.00049	0.00048	0.00048	0.00047	0.00047	0.00046	0.00046
7.7	0.00045	0.00045	0.00044	0.00044	0.00044	0.00043	0.00043	0.00042	0.00042	0.00041
7.8	0.00041	0.00041	0.00040	0.00040	0.00039	0.00039	0.00039	0.00038	0.00038	0.00037
7.9	0.00037	0.00037	0.00036	0.00036	0.00036	0.00035	0.00035	0.00035	0.00034	0.00034
8.0	0.00034	0.00033	0.00033	0.00033	0.00032	0.00032	0.00032	0.00031	0.00031	0.00031
8.1	0.00030	0.00030	0.00030	0.00029	0.00029	0.00029	0.00029	0.00028	0.00028	0.00028
8.2	0.00027	0.00027	0.00027	0.00027	0.00026	0.00026	0.00026	0.00026	0.00025	0.0025
8.3	0.00025	0.00025	0.00024	0.00024	0.00024	0.00024	0.00023	0.00023	0.00023	0.00023
8.4	0.00022	0.00022	0.00022	0.00022	0.00022	0.00021	0.00021	0.00021	0.00021	0.00021
8.5	0.00020	0.00020	0.00020	0.00020	0.00020	0.00019	0.00019	0.00019	0.00019	0.00019
8.6	0.00018	0.00018	0.00018	0.00018	0.00018	0.00018	0.00017	0.00017	0.00017	0.00017
8.7	0.00017	0.00016	0.00016	0.00016	0.00016	0.00016	0.00016	0.00016	0.00015	0.00015
8.8	0.00015	0.00015	0.00015	0.00015	0.00014	0.00014	0.00014	0.00014	0.00014	0.00014
8.9	0.00014	0.00014	0.00013	0.00013	0.00013	0.00013	0.00013	0.00013	0.00013	0.00012
9.0	0.00012	0.00012	0.00012	0.00012	0.00012	0.00012	0.00012	0.00012	0.00011	0.00011
9.1	0.00011	0.00011	0.00011	0.00011	0.00011	0.00011	0.00011	0.00010	0.00010	0.00010
9.2	0.00010	0.00010	0.00010	0.00010	0.00010	0.00010	0.00010	0.00009	0.00009	0.00009
9.3	0.00009	0.00009	0.00009	0.00009	0.00009	0.00009	0.00009	0.00009	0.00008	0.00008
9.4	0.00008	0.00008	0.00008	0.00008	0.00008	0.00008	0.00008	0.00008	0.00008	0.00008
9.5	0.00007	0.00007	0.00007	0.00007	0.00007	0.00007	0.00007	0.00007	0.00007	0.00007
9.6	0.00007	0.00007	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00006
9.7	0.00006	0.00006	0.00006	0.00006	0.00006	0.00006	0.00006	0.00006	0.00006	0.00006
9.8	0.00006	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005
9.9	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005	0.00005

Appendix Table 5-A

Single-Payment Compound Amount Factor (CAF)

Number of years (n)	Annual interest rates						
	3%	4%	5%	6%	7%	8%	10%
1	1.030	1.040	1.050	1.060	1.070	1.080	1.100
2	1.001	1.082	2.103	1.124	1.145	1.166	1.210
3	1.093	1.125	1.158	1.191	1.225	1.260	1.331
4	1.126	1.170	1.216	2.262	1.311	1.360	1.464
5	1.159	1.217	1.276	2.338	1.403	1.469	1.611
6	1.194	1.265	1.340	1.419	1.501	1.587	1.772
7	1.210	1.316	1.407	1.504	1.606	1.714	1.949
8	1.267	1.369	1.477	1.594	1.718	1.851	2.144
9	1.305	1.423	1.551	1.689	1.838	1.999	2.358
10	1.344	1.480	1.629	1.791	1.967	2.159	2.594
11	1.384	1.530	1.710	1.898	2.105	2.332	2.853
12	1.426	1.601	1.706	2.012	2.252	2.518	3.138
13	1.469	1.665	1.886	2.133	2.410	2.720	3.452
14	1.513	1.732	1.980	2.261	2.579	2.937	3.797
15	1.558	1.801	2.079	2.397	2.759	3.172	4.177
16	1.603	1.874	2.183	2.540	2.952	3.426	4.595
17	1.653	1.948	2.292	2.693	3.159	3.700	5.054
18	1.702	2.026	2.407	2.854	3.380	3.996	5.580
19	1.754	2.107	2.527	3.026	3.617	4.316	6.116
20	1.800	2.191	2.653	3.207	3.870	4.661	6.727
21	1.860	2.279	2.785	3.400	4.141	5.034	7.400
22	1.916	2.370	2.925	3.604	4.430	5.437	8.140
23	1.974	2.465	3.072	3.820	4.741	5.871	9.954
24	2.033	2.563	3.225	4.049	5.072	6.341	9.850
25	2.094	2.666	3.386	4.292	5.427	6.848	10.835

Appendix Table 5-B

Single-Payment Present Worth Factor (PWF)

Number of periods (n)	Interest rate of per period (i)							
	4%	5%	6%	7%	8%	10%	12%	15%
1	0.9615	0.9524	0.9434	0.9346	0.9259	0.9091	0.8929	0.8696
2	0.9248	0.9070	0.8800	0.8734	0.8573	0.8264	0.7972	0.7561
3	0.8890	0.8638	0.8398	0.8163	0.7938	0.7513	0.7118	0.6575
4	0.8548	0.8227	0.7921	0.7629	0.7350	0.6830	0.6355	0.5718
5	0.8219	0.7835	0.7473	0.7130	0.6806	0.6209	0.5674	0.4972
6	0.7903	0.7462	0.7050	0.6663	0.6302	0.5645	0.5066	0.4323
7	0.7599	0.7107	0.6651	0.6227	0.5835	0.5132	0.4523	0.3759
8	0.7307	0.6763	0.6274	0.5820	0.5403	0.4665	0.4039	0.3269
9	0.7026	0.6446	0.5919	0.5439	0.5002	0.4241	0.3606	0.2843
10	0.6756	0.6139	0.5584	0.5083	0.4632	0.3855	0.3220	0.2472
11	0.6498	0.5847	0.5268	0.4751	0.4289	0.3505	0.2875	0.2149
12	0.8246	0.5568	0.4970	0.4440	0.3971	0.3186	0.2567	0.1869
13	0.6006	0.5308	0.4688	0.4150	0.3677	0.2897	0.2292	0.1625
14	0.5775	0.5031	0.4423	0.3878	0.3405	0.2633	0.2046	0.1413
15	0.5553	0.4810	0.4173	0.3624	0.3152	0.2394	0.1827	0.1229
16	0.5339	0.4581	0.3936	0.3387	0.2919	0.2176	0.1631	0.1069
17	0.5134	0.4363	0.3714	0.3166	0.2703	0.1978	0.1456	0.0920
18	0.4936	0.4135	0.3503	0.2959	0.2502	0.1799	0.1300	0.0808
19	0.4746	0.8957	0.3305	0.2765	0.2317	0.1635	0.1161	0.0703
20	0.4564	0.8769	0.3118	0.2584	0.2145	0.1486	0.1037	0.0611
21	0.4388	0.3589	0.2942	0.2415	0.1987	0.1351	0.0926	0.0531
22	0.4220	0.3418	0.2775	0.2257	0.1839	0.1228	0.0826	0.0462
23	0.4067	0.3256	0.2618	0.2109	0.1703	0.1117	0.0738	0.0402
24	0.3901	0.3101	0.2470	0.1971	0.1577	0.1015	0.0659	0.0349
25	0.3751	0.2953	0.2330	0.1842	0.1460	0.0923	0.0588	0.0304
30	0.3083	0.2314	0.1741	0.1314	0.0994	0.0573	0.0334	0.0151
35	0.2534	0.1813	0.1301	0.0937	0.0676	0.0356	0.0189	0.0075
40	0.2088	0.1420	0.0972	0.0668	0.0460	0.0221	0.0107	0.0037
45	0.1712	0.1113	0.0727	0.0476	0.0313	0.0187	0.0061	0.0019
50	0.1407	0.0872	0.0543	0.0339	0.0213	0.0085	0.0035	0.0009

Appendix Table 5-C
Uniform Annual Series Compound Factor (CAFS)

Number of years (n)	Annual interest rates						
	3%	4%	5%	6%	7%	8%	10%
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	2.030	2.040	2.050	2.060	2.070	2.080	2.100
3	3.091	3.122	3.153	3.184	3.215	3.246	3.310
4	4.184	4.246	4.510	4.375	4.440	4.506	4.641
5	5.309	5.416	5.526	5.637	5.751	5.867	6.105
6	6.468	6.633	6.882	6.975	7.153	7.336	7.716
7	7.662	6.633	6.882	6.975	7.153	7.336	7.716
8	8.892	9.214	9.549	9.897	10260	10.537	11.436
9	10.159	10.583	11.927	11.491	11.978	12.488	13.579
10	11.464	12.006	12.578	13181	13.816	14.487	15.937
11	12.808	13.486	14.207	14.972	15784	16645	18.531
12	14.192	15.026	15.917	16.870	17.888	18.977	21.384
13	15.618	16.627	17.713	18.882	20.141	21.495	24.523
14	17.086	18.292	19.599	21.015	22.550	24.215	27975
15	18.599	20.024	21.579	23.276	25.129	27152	31.772
16	20.157	21.825	23.657	25.673	27.888	30.324	35.950
17	21.762	23.698	25.840	28213	30.840	33.750	40.545
18	23.414	25.645	28.132	30.906	33.999	37.450	45.599
19	25.117	27.671	30.539	33.760	37.379	41.446	51.159
20	26.870	29.778	33.066	36.786	40.995	45.762	57.275
21	28.676	31.969	35.719	39.993	44.865	50.423	64.002
22	30.537	34.248	38.505	43.392	49.006	55.457	71.403
23	32.453	36.618	41.430	46.996	53.436	60.893	79.543
24	34.426	39.083	44.502	50.816	58.177	66.765	88.497
25	36.459	41.646	47.727	54.865	63.249	73.106	98.347

Appendix Table 5-D

Uniform Annual Series Sinking Fund Factor (SFF)

Number of years (n)	Annual interest rates						
	3%	4%	5%	6%	7%	8%	10%
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.49261	0.49020	0.48780	0.48544	0.48309	0.48077	0.47169
3	0.32353	0.32035	0.317221	0.31411	0.31105	0.30803	0.30211
4	0.23903	0.23549	0.23201	0.22859	0.22523	0.23192	0.21547
5	0.18835	0.18463	0.18097	0.17740	0.17389	0.17046	0.16380
6	0.15460	0.15076	0.14702	0.14336	0.13980	0.13632	0.12961
7	0.13051	0.12661	0.12282	0.11914	0.11555	0.11207	0.10541
8	0.11246	0.10853	0.10472	0.10104	0.09747	0.09401	0.08744
9	0.09843	0.09449	0.09069	0.08702	0.08349	0.08008	0.07364
10	0.08723	0.08329	0.07950	0.07587	0.07238	0.06903	0.06275
11	0.07808	0.07415	0.07039	0.06679	0.06336	0.06008	0.05396
12	0.07046	0.06655	0.06283	0.05928	0.05590	0.05270	0.04676
13	0.064-3	0.06014	0.05646	0.05296	0.04965	0.04652	0.04078
14	0.05853	0.05467	0.05102	0.04758	0.04434	0.04130	0.03575
15	0.05377	0.04994	0.04634	0.04296	0.03979	0.03683	0.03147
16	0.04961	0.04582	0.04227	0.03895	0.03586	0.03298	0.02782
17	0.04595	0.04220	0.03870	0.03544	0.03248	0.02963	0.02446
18	0.04271	0.03899	0.03555	0.03236	0.02941	0.02670	0.02193
19	0.03981	0.03614	0.03275	0.02962	0.02675	0.02413	0.01955
20	0.03722	0.03558	0.03024	0.02718	0.02439	0.02185	0.01746
21	0.03487	0.03128	0.02800	0.02500	0.02229	0.01983	0.01562
22	0.03275	0.02920	0.02597	0.02305	0.02041	0.01803	0.01401
23	0.03081	0.02731	0.02414	0.02128	0.01871	0.01642	0.01257
24	0.02905	0.02559	0.02247	0.01968	0.01719	0.01498	0.01130
25	0.02743	0.02401	0.02095	0.01823	0.01581	0.01368	0.01017

Appendix Table 5-E

Capital Recovery Factor (CRF)

Numberof periods (n)	Interest rate of per period (i)							
	4%	5%	6%	7%	8%	10%	12%	15%
1	1.04000	1.05000	1.06000	1.07000	1.08000	1.10000	1.12000	1.13000
2	0.53020	0.53780	0.54544	0.55309	0.56077	0.57619	0.29170	0.61613
3	0.36035	0.36721	0.37411	0.38105	0.38803	0.40211	0.41635	0.43798
4	0.27549	0.28201	0.28859	0.29523	0.30192	0.31547	0.32923	0.35027
5	0.22463	0.23097	0.23740	0.24389	0.25046	0.26380	0.27741	0.29832
6	0.19076	0.19702	0.20336	0.20980	0.21632	0.22961	0.24323	0.26424
7	0.16661	0.17282	0.17914	0.18555	0.19207	0.20541	0.21912	0.24038
8	0.14853	0.15472	0.16104	0.16747	0.17401	0.18744	0.20130	0.22285
9	0.13449	0.14069	0.14702	0.15340	0.16008	0.17364	0.18768	0.20957
10	0.12329	0.12950	0.13587	0.14238	0.14903	0.16275	0.17698	0.19925
11	0.11415	0.12039	0.12679	0.13336	0.14008	0.15396	0.16842	0.19107
12	0.10655	0.11283	0.11928	0.12590	0.13270	0.14676	0.16144	0.18448
13	0.10014	0.10646	0.11296	0.11965	0.12652	0.14078	0.15568	0.17911
14	0.09467	0.10102	0.10758	0.11434	0.12130	0.13575	0.15087	0.17469
15	0.08994	0.09634	0.10296	0.10979	0.11683	0.13147	0.14632	0.17102
16	0.08582	0.09227	0.09895	0.10586	0.11298	0.12782	0.14339	0.16795
17	0.08220	0.08870	0.09544	0.10243	0.10963	0.12766	0.14046	0.16537
18	0.07899	0.08555	0.09236	0.09941	0010670	0.12193	0.13794	0.16319
19	0.07014	0.08275	0.08962	0.09675	0.10413	0.11955	0.13576	0.16134
20	0.07358	0.08024	0.08718	0.09439	0.10185	0.11746	0.13388	0.15976
21	0.07128	0.07800	0.08500	0.09220	0.09983	0.11562	0.13224	0.15842
22	0.06920	0.07597	0.08305	0.09041	0.09803	0.11401	0.13081	0.15727
23	0.06731	0.07414	0.08128	0.08871	0.09842	0.11257	0.12958	0.15626
24	0.06559	0.07247	0.07668	0.08719	0.09498	0.11130	0.12948	0.15543
25	0.06401	0.07095	0.07823	0.08581	0.09368	0.11070	0.12780	0.15470
30	0.5783	0.6505	0.07265	0.08059	0.08883	0.10608	0.12414	0.15230
35	0.05358	0.06107	0.06897	0.07723	0.08580	0.10369	0.12232	0.15113
40	0.05052	0.05828	0.06646	0.07501	0.08386	0.10223	0.12130	0.15066
45	0.04826	0.05626	0.06470	0.07350	0.08259	0.10139	0.12074	0.15026
50	0.04655	0.05478	0.06346	0.07246	0.08174	0.10089	0.12042	0.15014

Appendix Table 5-F

Uniform Annual Series Present Worth Foctor (PWFS)

Number of periods (n)	Interest rate of per period (i)							
	4%	5%	6%	7%	8%	10%	12%	15%
1	0.962	0.952	0.943	0.935	0.926	0.909	0.893	0.870
2	1.886	1.859	1.833	1.808	1.783	1.736	1.690	1.626
3	2.775	2.723	2.673	2.624	2.577	2.487	2.402	2.288
4	3.630	3.546	3.465	3.387	3.132	3.170	3.037	3.855
5	4.452	4.329	4.212	4.100	3.993	3.791	3.605	3.352
6	5.242	5.076	4.917	4.767	4.623	4.355	4.111	3.784
7	6.002	5.786	5.582	5.389	5.206	4.86 8	4.564	4.160
8	6.733	6.463	6.210	5.971	5.747	5.335	4.968	4.487
9	7.435	7.108	6.802	6.515	6.247	5.759	5.328	4.772
10	8.111	7.722	7.360	7.024	6.710	6.144	5.550	5.019
11	8.760	8.306	7.887	7.499	7.139	6.495	5.938	5.234
12	9.385	8.863	8.384	7.943	7.536	6.814	6.104	5.421
13	9.986	9.394	8.853	8.358	7.904	7.103	6.424	5.583
14	10.563	9.899	9.295	8.745	8.244	7.367	6.628	5.724
15	11.118	10.380	9.712	9.108	8.559	7.606	6.811	5.847
16	11.652	10.838	10.106	9.447	8.851	7.824	6.974	5.954
17	12.166	11.274	10.477	9.763	9.122	8.022	7.120	6.047
18	12.659	11.690	10.828	10.059	9.372	8.201	7.250	6.128
19	13.134	12.085	11.158	10.336	9.604	8.365	7.366	6.198
20	13.590	12.462	11.470	10.594	9.818	8.514	7.479	6.259
21	14.029	12.821	11.764	10.836	10.017	8.649	7.562	6.312
22	14.451	13.163	12.042	11.061	10.201	8.772	7.645	6.359
23	14.857	13.489	12.303	11.272	10.371	8.883	7.718	6.399
24	15.247	13.799	12.550	11.469	10.529	8.985	7.784	6.434
25	15.622	14.094	12.783	11.654	10.675	9.077	7.843	6.464
30	17.292	15.272	13.765	12.409	11.258	9.247	8.055	6.566
35	18.665	16.374	14.498	12.948	11.655	9.644	8.176	6.617
40	19.793	17.159	15.046	13.332	11.925	9.779	8.244	6.642
45	20.720	17.774	15.456	13.606	12.108	9.863	8.288	6.654
50	21.482	18.256	15.762	13.801	12.233	9.915	8.305	6.661

Appendix Table 6-A

Table of Random Numbers

39 65 76 45 45	19 90 69 64 61	20 26 36 31 62	58 24 57 14 97	95 06 70 99 00
73 71 23 70 90	65 97 60 12 11	31 56 34 19 19	47 83 75 51 33	30 62 38 20 44
72 20 47 33 84	51 67 47 97 19	98 40 07 17 66	23 05 09 51 80	59 7811 52 69
75 17 25 69 17	17 95 21 78 58	24 33 45 77 48	69 81 84 09 29	93 22 70 45 80
37 48 79 88 74	63 52 06 34 30	01 31 60 10 27	35 07 79 71 53	28 99 52 01 64
02 89 08 16 94	85 53 83 29 95	56 27 09 24 43	21 78 55 09 82	72 61 88 73 61
87 18 15 70 07	37 79 49 12 38	48 13 93 15 96	41 92 45 71 51	09 18 25 58 94
98 83 71 70 15	89 09 39 59 24	00 06 41 14 20	14 36 59 25 47	54 45 17 24 89
10 08 58 07 04	76 62 16 48 68	58 76 17 14 86	59 53 11 52 21	66 04 18 7287
47 90 56 37 31	71 82 13 50 14	27 55 10 24 92	28 04 67 53 44	95 23 00 84 47
93 05 31 03 07	34 18 04 52 35	74 13 39 35 22	68 95 23 92 35	36 63 70 35 31
21 89 11 47 99	11 20 99 45 18	76 51 94 84 86	13 79 93 37 55	98 16 04 41 67
95 18 94 36 97	27 37 83 28 71	79 57 95 13 91	09 61 87 25 21	56 11 20 32 44
97 08 31 55 73	10 65 81 92 59	77 31 61 95 46	20 44 90 32 64	26 99 76 75 63
69 26 88 86 13	59 71 74 17 32	48 38 75 93 29	73 37 32 04 05	60 82 29 20 25
41 47 10 25 03	87 63 93 95 17	81 83 83 04 49	77 45 85 50 51	79 88 01 97 30
91 94 15 63 62	08 61 74 51 68	92 79 43 83 79	29 18 94 51 23	14 85 11 47 23
80 06 54 18 47	08 52 85 08 40	48 40 35 94 22	72 65 71 08 86	50 03 42 99 36
76 72 77 63 99	89 85 84 46 06	64 71 06 21 66	89 37 20 70 01	61 65 70 22 12
59 40 24 13 75	42 29 82 23 19	06 94 79 10 08	81 30 15 39 14	81 83 17 16 33
63 62 06 34 41	79 53 36 02 95	94 61 09 43 62	20 21 14 68 86	84 95 48 46 45
78 47 23 53 90	79 93 96 38 63	34 85 52 05 09	84 43 01 72 73	14 93 87 81 40
87 68 62 15 43	97 48 72 66 48	53 16 71 13 81	59 97 50 99 52	24 62 20 42 31
47 60 92 10 77	26 97 05 73 51	88 46 38 00 58	72 68 49 29 31	75 70 16 08 24
56 88 87 59 41	06 87 37 78 48	65 88 69 58 39	88 02 84 27 83	85 81 56 59 38
22 17 68 65 84	86 02 22 57 51	68 69 8095 44	11 29 01 95 80	49 34 35 96 47
19 36 27 59 46	39 77 32 77 09	79 57 92 36 39	89 74 39 82 15	08 50 94 34 74
16 77 23 02 77	28 06 24 25 93	22 45 44 84 11	87 80 61 65 31	09 71 91 74 25
78 43 66 71 61	97 66 63 99 61	80 45 67 93 82	59 73 19 85 23	53 33 65 97 21
03 28 28 26 08	69 30 16 09 05	53 58 47 70 93	66 56 45 63 79	45 56 20 19 47
04 31 17 21 56	33 63 99 19 87	26 72 39 27 67	53 77 57 69 93	60 61 97 22 61
61 06 98 03 91	87 14 77 43 96	43 00 65 98 50	45 60 33 01 07	98 99 46 50 47
23 58 35 26 00	99 53 93 61 28	52 70 05 48 34	56 65 05 61 86	90 92 10 78 80
15 39 25 70 99	93 86 52 77 65	15 35 59 05 28	22 87 26 07 47	86 96 98 29 06
58 71 96 30 24	18 46 23 34 27	85 13 99 24 44	49 18 09 79 49	74 16 32 23 02
93 22 53 64 39	07 10 63 76 35	87 03 04 79 88	08 33 33 85 51	55 34 57 72 69
78 76 58 54 74	92 38 70 96 92	52 06 79 79 45	82 63 18 27 44	69 66 92 19 09
61 81 31 96 82	00 57 25 60 29	46 72 60 18 77	55 66 12 62 11	08 99 55 64 57
42 88 31 96 82	24 98 65 08 21	47 21 61 88 32	27 80 30 21 60	10 92 35 64 57
77 94 30 05 33	28 10 99 00 27	12 73 73 99 12	39 99 57 94 82	96 88 57 17 91

Appendix Table 6-B

Table of Random Numbers (Continued)

8481	5016	0080	4376	2579	8293	5950	1048	0650	4135
0744	3447	6173	3288	6378	6704	0966	9986	5202	0728
5558	7239	2976	4836	6134	5120	1541	6514	3581	2079
9371	1463	2164	2301	3142	3866	8707	9988	2011	5111
3033	1660	6365	9054	1155	8844	4085	9589	2924	1725
1053	7320	6532	7214	8972	6466	1217	0100	1458	9416
4309	3504	4086	9434	0136	5965	6876	7937	5476	3396
2158	8854	9534	1196	4941	2697	7497	1149	1752	3482
6749	3676	4943	1406	8614	2060	6433	1660	8875	3194
2878	3447	4804	6761	5309	0636	0522	2004	3207	4684
0591	6549	2206	6185	6188	2649	2389	9483	0924	1389
1025	3438	0546	2545	1089	1280	6701	9742	3453	5573
4244	9217	1623	4524	0163	9895	9586	2083	8459	0644
1331	9032	1388	5661	0472	7128	1902	0343	7724	6528
8853	3490	2589	8744	1221	4667	8396	4779	9937	7206
5059	4192	6331	5485	5922	0982	9390	8993	3621	2602
0821	4340	3194	0118	4773	1891	1891	7989	9190	2296
5262	1746	7108	6496	2570	5029	5029	8949	4989	5008
1210	1858	9365	6562	0269	1776	1796	6625	8591	1990
3642	6629	5775	3219	8801	6861	6861	0765	2379	3494
9598	5322	3747	0363	5995	5504	6804	7033	0957	9516
3894	3173	2853	9312	2498	8878	4956	8748	6247	0673
3603	3011	6762	0848	8316	3485	6388	8925	3799	0898
1121	2978	6313	5857	8457	1395	7240	8630	3896	6348
1930	4583	4227	4120	6893	7005	2264	6067	5627	7985
6309	9158	2830	3262	9809	4606	8869	1454	5841	7696
4460	3143	5383	0327	9668	1697	8335	0869	2188	1908
8371	5095	7273	1866	4193	4163	2035	2832	4996	7143
9371	5549	9298	9076	1299	6669	2088	2809	9631	3162
9304	1468	4013	7465	0861	6787	3581	7977	8409	4798
5606	2435	8546	3209	4802	6690	8527	2219	6706	1930
6693	8333	8082	7546	2910	853	8725	1237	4423	1570
0556	7715	8994	4245	1540	8159	3889	5273	6977	2703
6973	9299	4959	7146	1426	7086	8743	6982	5547	3394
4920	1223	5208	6661	4907	1102	0501	3625	8513	3192
0132	0928	8241	0858	7627	4174	1170	3142	2455	4891
4051	3101	9854	4488	6931	3266	3147	2560	8011	8848
0267	5612	5504	7917	7928	8034	9989	4351	2075	9497
0609	9469	3149	4086	8911	8547	3518	9349	1836	0548
2593	1666	3750	5105	4287	4380	7860	7792	1625	7659
8812	9491	2602	4100	7962	1037	9778	1778	4223	3193
3540	5985	0019	7155	1471	1851	8682	9957	3772	4706
9335	5375	1239	1624	5378	6803	7177	7911	4660	5669
3174	7677	8282	6669	0879	7874	9931	6581	9784	2697
8864	4760	1129	6205	4949	4105	0222	7479	6470	8194
5245	7641	0593	5656	6799	3071	1751	4339	5630	9496
5468	6083	4511	1440	2135	5777	9903	1048	6726	8602
3951	7928	6818	4161	4840	1392	1323	5014	7558	9854
7319	4064	4024	5401	2834	7518	3978	3742	1005	4619
5892	8731	6269	5189	2071	4084	9789	3620	9819	4548

Appendix Table 6-C

Table of Random Numbers (Continued)

09 73 25 33	76 53 01 35 86	34 67 35 48 76	80 95 90 90 17	39 29 27 49
54 20 48 05	64 89 47 72 96	24 80 52 40 37	20 63 61 04 02	00 82 29 16
42 26 89 53	19 64 50 93 03	23 20 9025 60	15 95 33 47 64	35 08 03 36
01 90 25 29	09 37 67 07 15	38 31 13 11 65	88 67 67 43 97	04 43 62 76
80 79 99 70	80 15 73 61 47	64 03 23 66 53	98 95 11 68 77	12 17 17 68
06 57 47 17	34 07 27 68 50	36 69 73 61 70	65 81 33 98 85	11 19 92 91
06 01 08 05	47 57 1824 06	35 30 34 26 14	86 79 90 74 99	23 40 30 97
26 97 76 02	02 05 16 56 92	68 66 5748 18	73 05 38 52 47	18 62 38 85
57 33 21 35	05 32 54 7048	90 55 35 75 48	23 46 82 87 09	82 49 12 56
79 64 57 53	03 52 96 47 78	35 80 83 42 82	60 93 52 03 44	35 27 38 84
52 01 77 67	14 90 56 86 07	22 10 94 05 58	60 97 09 34 33	50 50 07 39
80 50 54 31	39 80 82 77 32	50 72 56 82 48	29 40 52 42 01	52 77 56 78
45 29 96 34	06 28 89 80 83	13 74 67 00 78	18 47 54 06 10	68 71 17 78
68 34 02 00	86 50 85 84 01	36 76 66 79 51	90 36 47 64 93	29 80 91 01
59 46 73 48	87 51 76 49 69	91 82 60 89 28	93 78 56 13 68	23 47 83 41
48 11 76 74	17 46 85 09 50	58 04 77 69 74	73 03 95 71 86	40 21 81 65
12 43 56 35	17 72 70 80 15	45 31 92 23 74	21 11 57 82 53	14 38 55 37
35 09 98 17	77 40 27 72 14	43 23 60 02 10	45 52 16 42 37	96 28 60 26
91 92 68 03	66 25 22 91 48	36 93 68 72 03	76 62 11 39 90	94 40 05 64
89 32 05 05	14 22 56 85 14	46 42 75 67 88	96 29 77 88 22	54 38 21 45
49 91 45 23	68 47 92 76 86	46 16 28 35 54	94 75 08 99 23	37 08 92 00
33 69 45 98	26 94 03 68 58	70 29 73 41 35	53 14 03 33 40	45 05 08 23
10 48 19 49	85 15 74 79 54	32 97 92 65 75	57 60 04 08 81	22 22 20 64
55 07 37 42	11 10 00 20 40	12 86 07 46 97	96 64 48 94 39	28 70 72 58
60 64 93 29	16 50 53 44 84	40 21 95 25 63	43 65 17 70 82	07 20 73 17
19 69 04 46	26 45 74 77 74	51 92 43 37 29	65 39 45 95 93	42 58 26 05
47 44 52 66	95 27 07 99 53	59 36 78 38 48	82 39 61 01 18	33 21 15 94
55 72 85 83	67 89 75 43 87	54 62 24 44 31	91 19 04 25 92	92 92 74 59
48 11 62 13	97 34 40 87 21	16 86 84 87 67	02 07 11 20 59	25 70 24 66
52 37 83 17	73 20 88 98 37	68 93 59 14 16	26 25 22 96 63	05 52 28 25
49 35 24 94	75 24 63 38 24	45 86 25 10 25	61 96 27 93 35	65 33 71 24
54 99 76 54	64 05 18 81 59	96 11 96 38 96	54 69 28 23 91	23 28 72 95
96 32 53 07	26 89 80 93 54	33 35 13 54 62	77 97 45 00 24	90 10 33 93
80 80 83 91	45 42 72 68 42	83 60 94 97 00	13 02 12 48 92	78 56 52 01
05 88 52 36	01 39 09 22 86	77 28 14 40 77	93 91 08 36 47	70 61 74 29
17 90 02 97	87 87 92 52 41	05 56 70 70 07	86 74 31 71 57	85 39 41 18
23 46 14 06	20 11 74 52 04	15 95 66 00 00	18 74 39 24 23	97 11 89 63
56 54 14 30	01 75 87 53 79	40 41 92 15 85	66 67 43 68 06	84 96 28 52
15 51 49 38	19 47 60 72 46	43 66 79 45 43	59 04 79 00 33	20 82 66 85
86 43 19 94	36 16 81 08 51	34 88 88 15 53	01 54 03 54 56	05 01 45 11
08 62 48 26	45 24 02 84 04	44 99 90 88 96	39 09 47 84 07	35 44 13 18
18 51 62 32	41 94 15 09 49	89 43 54 85 81	88 69 54 19 94	37 54 87 30
0510 04 06	96 33 27 07 74	2015 12 33 87	25 01 62 52 98	94 62 46 11

Appendix Table 6-D

Table of Random Numbers (Continued)

218122396	2068577984	8262130892	8374856049	4837657422
1128105582	7295088579	9586111652	7055508767	5172382962
7112077556	3440672486	1882412963	0684012006	2290031331
6575477468	5435810788	9670852913	1291295730	2290031331
4199520858	3090908872	2039593181	5973470495	8076135523
3545174840	2275698645	8416549348	4676463101	5629367907
1749420382	4832630032	5670984959	5432114610	0666095693
9103161011	7413686599	1198757695	0414294470	9240121544
0764238934	7666127259	5263097712	5133648980	5119669120
3493969525	0272759769	0385998136	9999089966	1344056826
1292054466	0700014629	5169349659	8408705169	6574373193
4397426117	6488888550	4031652526	8123543276	6027534501
3807950579	9564268448	3457416988	1531027886	5116633717
4984768758	2389278610	3859431781	3643768456	5041314549
1340145867	9120831830	7228567652	1267173884	13206751658
0590453442	4800088084	1165628554	5407921254	9468932498
9566554338	5585265145	5089052204	9780623691	5795448061
7615116284	9172824179	5544814334	0016943666	2628538741
8508771938	4035554324	0840126299	4942059208	7875623913
6070024586	9324732696	1186263397	4425143189	3316653259
5799997185	0135968939	7678931194	1351031403	6002561840
6364375912	8383232768	1892850701	2323673751	3188881718
4165492027	6349104233	3382569662	4579426926	1313082455
0354683246	4765104877	8149224168	5468631609	6774393896
9130555058	5255147182	3519287786	2481675649	8907598697
5826984369	4725370390	9681916289	5049082870	7463807244
6285048453	3646121751	8436077768	2928794356	9956043516
7527791048	5765558107	8762562043	6185670830	6363845920
8976470693	0441608934	8749472723	2202271078	5897002653
2327991661	7936797054	9527542791	4711871173	8300978148
1182095589	5535798279	4764439855	6279247618	4446895088
3659397698	1056981450	8416606706	8234013222	6426813469
5924779358	1333750468	9434074212	5273692238	5902177065
3941102295	5726289716	3420847871	1820481234	0318831723
1955104281	0903099163	6827824899	6383872737	5901682626
2117595534	1634107293	8521057422	1471300754	3044151557
7471564123	7344613447	1128117244	3208461091	1699403490
8674262892	2809456764	5806554509	8224980942	5738031833
9061122871	0746980892	9285305274	6331989649	8764467686
6438538678	3049068967	6955157269	5482964330	2161984904
1834182305	6203476893	5937802079	3445280195	3694915658
1884227732	2923727501	8044389132	4611203081	607211445
6791857341	6696243386	2219599137	3193884246	8224729918

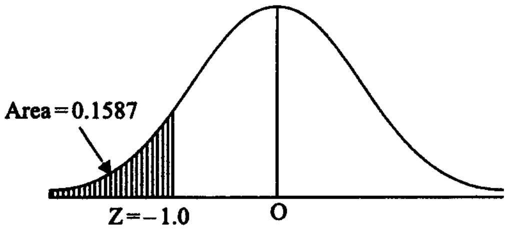
Appendix Table 7

Random normal numbers ($\mu = 0$, $\sigma = 1$)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.464	0.137	2.455	-0.323	-0.068	0.296	-0.288
2	0.060	-2.526	-0.531	-1.940	0.543	-1.558	0.187
3	1.486	-0.354	-0.634	0.697	0.926	1.375	0.785
4	1.022	-0.472	1.279	3.521	0.571	-1.851	0.194
5	1.394	-0.555	0.046	0.321	2.945	1.974	-0.258
6	0.906	-0.513	-0.525	0.595	0.881	-0.934	1.579
7	1.179	-1.055	0.007	0.769	0.971	0.712	1.090
8	-1.501	-0.488	-0.162	-0.136	1.003	0.203	0.448
9	-0.690	0.756	-1.618	-0.445	-0.511	-2.051	-0.457
10	1.372	0.225	0.378	0.761	0.181	-0.736	0.960
11	-0.482	1.677	-0.057	-1.229	-0.486	0.856	-0.491
12	-1.376	-0.150	1.356	-0.561	-0.256	0.212	0.219
13	-1.010	0.598	-0918	1.598	1.065	0.415	-0.169
14	-0.005	-0.899	0.012	-0.725	1.147	-0.121	-0.196
15	1.393	-1.163	-0.911	1.231	-0.199	-0.246	1.239
16	-1.787	-0.261	1.237	1.046	-0.508	-1.630	-0.146
17	-0.105	-0.357	-1.384	0.360	-0.992	-0.110	-1.698
18	-1.339	1.827	-0.959	0.421	0.969	-1.141	-1.041
19	1.041	0.535	0.731	1.377	0.983	-1.330	1.620
20	-0.279	-2.056	0.717	-0.873	-1.096	-1.396	1.047
21	-1.805	-2.008	-1.633	0.542	0.250	0.166	0.032
22	-1.186	1.180	1.114	0.882	1.205	-0.202	0.151
23	0.658	-1.141	1.151	-1.210	-0.927	0.425	0.290
24	-0.439	0.358	-1.939	0.891	-0.227	0.602	0.973
25	1.398	-0.230	0.385	-0.649	-0.577	0.237	-0.289
26	0.199	0.208	-1.083	-0.219	-0.291	1.221	1.119
27	0.159	0.272	-0.313	0.084	-2.828	-0.439	-0.792
28	2.273	0.606	0.606	-0.747	0.247	1.291	0.063
29	0.041	-0.307	0.121	0.790	-0.584	0.541	0.484
30	-1.132	-2.098	0.921	0.145	0.446	-2.661	1.045
31	0.768	0.079	-1.473	0.034	-2.127	0.665	0.084
32	0.375	-1.658	-0.851	0.234	-0.656	0.340	-0.086
33	-0.513	-0.344	0.210	-1.736	1.041	0.008	0.427
34	0.292	-0.521	1.266	-1.206	-0.899	0.110	-0.528
35	1.026	2.990	-0.574	-0.491	-1.114	1.297	-1.433
36	-1.334	1.278	-0.568	-0.109	-0.515	-0.566	2.923
37	-0.287	-0.144	-0.254	0.574	-0.451	-1.181	-1.190
38	0.161	-0.886	-0.921	-0.509	1.410	-0.518	0.192
39	-1.346	0.193	-1.202	0.394	-1.045	0.843	0.942
40	1.250	-0.199	-0.288	1.810	1.378	0.584	1.216

Appendix Table 8-A

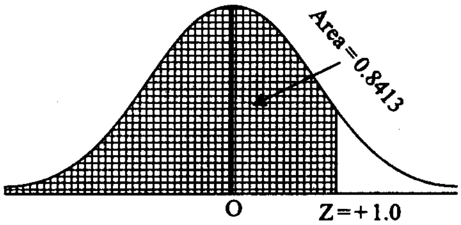
Area Under the Standard Normal Distribution for Negative Values of Z



Z to First Decimal	Seond Decimal									
	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
– 3.0	.0014	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
– 2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
– 2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
– 2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
– 2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
– 2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
– 2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
– 2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
– 2.2	.0139	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110
– 2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
– 2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
– 1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0238	.0233
– 1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0300	.0294
– 1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
– 1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
– 1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0570	.0559
– 1.4	.0808	.7938	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
– 1.3	.0968	.0951	.0934	.0918	.0901	.0855	.0869	.0853	.0838	.0823
– 1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
– 1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
– 1.0	.1587	.1562	.1529	.1515	.1492	.1469	.1446	.1423	.1401	.1379
– 0.9	.1841	.1814	.1785	.1762	.1736	.1711	.1685	.1660	.1635	.1611
– 0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
– 0.7	.2420	.2389	.2358	.2327	.2297	.2266	.2236	.2206	.2177	.2143
– 0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
– 0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
– 0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
– 0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
– 0.2	.4207	.4168	.4129	.4090	.4052	4013	.3974	.3936	.3897	.3859
– 0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
– 0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Appendix Table 8-B

Area Under the Standard Normal Distribution from extreme Left to Positive Values of Z



Example. To find the area under the curve from extreme left ($Z = -\infty$) to a point $Z = 1.0$ to the right of the mean, look up the value opposite 1.0 in the table; .8413 of the area under the curve lies between the extreme left ($z = -\infty$) to a z value.

Z to First Decimal	Second Decimal									
	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6841	.6879
0.5	.6915	.6950	.6985	.9019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7406	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8910	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9430	.9441
1.6	.9452	.9463	.9474	.9485	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9700	.9706
1.9	.9713	.9719	.9726	.9732	.9783	.9744	.9750	.9756	.9762	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9865	.9868	.9871	.9874	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9924	.9926	.9928	.9930	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9944	.9946	.9932	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9958	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9979	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9986	.9987	.9987	.9988	.9988	.9988	.9989	.9989	.9990	.9990

Appendix Table 9

Proportion of Total Area Under the Normal Curve From ∞ To t, Where $t = (x - mu)/\sigma$

t	ψ (t)	t	ψ (t)	t	ψ (t)	t	ψ (t)
0.00	0.5000	0.65	0.7422	1.30	0.9032	1.95	0.9744
0.01	0.5040	0.66	0.7454	1.31	0.9049	1.96	0.9750
0.02	0.5080	0.67	0.7486	1.32	0.9066	1.97	0.9756
0.03	0.5120	0.68	0.7517	1.33	0.9082	1.98	0.9761
0.04	0.5160	0.69	0.7549	1.34	0.9099	1.99	0.9767
0.05	0.5199	0.70	0.7580	1.35	0.9115	2.00	0.9772
0.06	0.5239	.071	0.7611	1.36	0.9131	2.02	0.9783
0.07	0.5279	0.72	0.7642	1.37	0.9147	2.04	.09793
0.08	0.5319	0.73	0.7673	1.38	0.9162	2.06	0.9803
0.09	0.5359	0.74	0.7703	1.39	0.9177	2.08	0.9812
0.10	0.5398	0.75	0.7734	1.40	0.9192	2.10	0.9821
0.11	0.5438	0.76	0.7764	1.41	0.9207	2.12	0.9830
0.12	0.5478	0.77	0.7794	1.42	0.9222	2.14	0.9838
0.13	0.5517	0.78	0.7823	1.43	0.9236	2.16	0.9846
0.14	0.5557	0.79	0.7852	1.44	0.9251	2.18	0.9854
0.15	0.5596	0.80	0.7881	1.45	0.9265	2.20	0.9861
0.16	0.5636	0.81	0.7910	1.46	0.9279	2.22	0.9868
0.17	0.5675	0.82	0.7939	1.47	0.9292	2.24	0.9875
0.18	0.5714	0.83	0.7967	1.48	0.9306	2.26	0.9881
0.19	0.5753	0.84	0.7995	1.49	0.9319	2.28	0.9887
0.20	0.5793	0.85	0.8023	1.50	0.9332	2.30	0.9893
0.21	0.5832	0.86	0.8051	1.51	0.9345	2.32	0.9898
0.22	0.5871	0.87	0.8078	1.52	0.9357	2.34	0.9904
0.23	0.5910	0.88	0.8106	1.53	0.9370	2.36	0.9909
0.24	0.5948	0.89	0.8133	1.54	0.9382	2.38	0.9913
0.25	0.5987	0.90	0.8159	1.55	0.9394	2.40	0.9918
0.26	0.6026	0.91	0.8186	1.56	0.9406	2.42	0.9922
0.27	0.6064	0.92	0.8212	1.57	0.9418	2.44	0.9927
0.28	0.6103	0.93	0.8238	1.58	0.9429	2.46	0.9931
0.29	0.6141	0.94	0.8264	1.59	0.9441	2.48	0.9934
0.30	0.6179	0.95	0.8289	1.60	0.9252	2.50	0.9938
0.31	0.6217	0.96	0.8315	1.61	0.9463	2.52	0.9941
0.32	0.6255	0.97	0.8340	1.62	0.9474	2.54	0.9945
0.33	0.6293	0.98	0.8365	1.63	0.9484	2.56	0.9948
0.34	0.6331	0.99	0.8389	1.64	0.9495	2.58	0.9951
0.35	0.6368	1.00	0.8413	1.65	0.9505	2.60	0.9953
0.36	0.6406	0.01	0.8438	1.66	0.9515	2.62	0.9956
0.37	0.6443	1.02	0.8461	1.67	0.9525	2.64	0.9959
0.38	0.6480	1.03	0.8485	1.68	0.9535	2.66	0.9961
0.39	0.6517	1.04	0.8508	1.69	0.9545	2.68	0.9963
0.40	0.6554	1.05	0.8531	1.70	0.9554	2.70	0.9965
0.41	0.6591	1.06	0.8554	1.71	0.9564	2.72	0.9967
0.42	0.6628	1.07	0.8577	1.72	0.9573	2.74	0.9969
0.43	0.6664	1.08	0.8599	1.73	0.9582	2.76	0.9971
0.44	0.6700	0.09	0.8621	1.74	0.9591	2.78	0.9973
0.45	0.6736	1.10	0.8643	1.75	0.5999	2.80	0.9974
0.46	0.6772	1.11	0.8665	1.76	0.9608	2.82	0.9976
0.47	0.6808	1.12	0.8686	1.77	0.9616	2.84	0.9977
0.48	0.6844	1.13	0.8708	1.78	0.9625	2.86	0.9979
0.49	0.6879	1.14	0.8729	1.79	0.9633	2.88	0.9980
0.50	0.6915	1.15	0.8749	1.80	0.9641	2.90	0.9981
0.51	0.6950	1.16	0.8770	1.81	0.9649	2.92	0.9982
0.52	0.6985	1.17	0.8790	1.82	0.9656	2.94	0.9984
0.53	0.7019	1.18	0.8810	1.83	0.9664	2.96	0.9985
0.54	0.7054	1.19	0.8830	1.84	0.9671	2.98	0.9986
0.55	0.7088	1.20	0.8849	1.85	0.9678	3.00	0.99865
0.56	0.7123	1.21	0.8869	1.86	0.9686	3.20	0.99931
0.57	0.7190	1.22	0.8888	1.87	0.9693	3.40	0.99966
0.58	0.7190	1.23	0.8907	1.88	0.9699	3.60	0.999841
0.59	0.7224	1.24	0.8925	1.89	0.9706	3.80	0.999928
0.60	0.7257	1.25	0.8944	1.90	0.9713	4.00	0.999968
0.61	0.7291	1.26	0.8962	1.91	0.9719	4.50	0.999997
0.62	0.7324	1.27	0.8980	1.92	0.9726	5.00	0.999997
0.63	0.7357	1.28	0.8997	1.93	0.9732		
0.64	0.7389	1.29	0.9015	1.94	0.9738		

Appendix Table 10

Values of ϵ^x and e^{-x}

x	e^x	e^{-x}	x	e^x	e^{-x}
0.00	1.000	1.000	3.00	20.086	0.050
0.10	1.105	0.905	3.10	22.198	0.045
0.20	1.221	0.819	3.20	24.533	0.041
0.30	1.350	0.741	3.30	27.113	0.037
0.40	1.492	0.670	3.40	29.964	0.033
0.50	1.649	0.607	3.50	33.115	0.030
0.60	1.822	0.549	3.60	36.598	0.027
0.70	2.014	0.497	3.70	40.447	0.025
0.80	2.226	0.449	3.80	44.701	0.022
0.90	2.460	0.407	3.90	49.402	0.020
1.00	2.718	0.368	4.00	54.598	0.018
1.10	3.004	0.333	4.10	60.340	0.017
1.20	3.320	0.301	4.20	66.686	0.015
1.30	3.669	0.273	4.30	73.700	0.014
1.40	4.055	0.247	4.40	81.451	0.012
1.50	4.482	0.223	4.50	90.017	0.011
1.60	4.953	0.202	4.60	99.484	0.010
1.70	5.474	0.183	4.70	109.95	0.009
1.80	6.050	0.165	4.80	121.51	0.008
1.90	6.686	0.150	4.90	134.29	0.007
2.00	7.389	0.135	5.00	148.41	0.007
2.10	8.166	0.122	5.10	164.02	0.066
2.20	9.025	0.111	5.20	181.27	0.066
2.30	9.974	0.100	5.30	200.34	0.055
240	11.023	0.091	5.40	221.41	0.055
2.50	12.182	0.082	5.50	244.69	0.004
2.60	13.464	0.074	5.60	270.43	0.004
2.70	14.880	0.067	5.70	298.87	0.003
2.80	16.445	0.061	5.80	330.30	0.003
2.90	18.174	0.055	590	365.04	0.003
3.00	20.086	0.050	6.00	403.43	0.002

Appendix Table 11-A

Poisson Distribution

x/λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	.9048	.8187	.7408	.6703	.6065	.5488	.4966	.4493	.4066	.3679
1	.0905	.1637	.2222	.2681	.3033	.3293	.3476	.3595	.3659	.3679
2	.0045	.0164	.0333	.0536	.0758	.0988	.1217	.1438	.1647	.1839
3	.0002	.0011	.0033	.0072	.0126	.0198	.0284	.0383	.0494	.0613
4	.0000	.0001	.0002	.0007	.0016	.0030	.0050	.0077	.0111	.0153
5	.0000	.0000	.0000	.0001	.0002	.0004	.0007	.0012	.0020	.0031
6	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0005
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

x/λ	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	.3329	.3012	.2725	.2466	.2231	.2019	.1827	.1653	.1496	.1353
1	.3662	.3014	.3543	.3452	.3347	.3230	.3106	.2957	.2842	.2707
2	.2014	.2169	.2303	.2417	.2510	.2584	.2640	.2678	.2700	.2707
3	.0738	.0867	.0998	.1128	.1255	.1378	.1496	.1607	.1710	.1804
4	.0203	.0260	.0324	.0395	.0471	.0551	.0636	.0723	.0812	.0902
5	.0045	.0062	.0084	.0111	.0141	.0176	.0216	.0260	.0309	.0361
6	.0008	.0012	.0018	.0026	.0035	.0047	.0061	.0078	.0098	.0120
7	.0001	.0002	.0003	.0005	.0008	.0011	.0015	.0020	.0027	.0034
8	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0006	.0009
9	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002

x/λ	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	.1225	.1108	.1003	.0907	.0821	.0743	.0672	.0608	.0550	.0498
1	.2572	.2438	.2306	.2177	.2052	.1931	.1815	.1703	.1596	.1494
2	.2700	.2681	.2652	.2613	.2505	.2510	.2450	.2384	.2314	.2240
3	.1890	.1966	.2033	.2090	.2138	.2176	.2205	.2225	.2237	.2240
4	.0992	.1082	.1169	.1254	.1336	.1414	.1488	.1557	.1622	.1680
5	.0417	.0476	.0538	.0602	.668	.0735	.0804	.0872	.0940	.1008
6	.0146	.0174	.0206	.0241	.0278	.0319	.0362	.0407	.0455	.0504
7	.0044	.0055	.0068	.0083	.0099	.0118	.0139	.0163	.0188	0.216
8	.0011	.0015	.0019	.0025	.0031	.0038	.0047	.0057	.0068	.0081
9	.0003	.0004	.0005	.0007	.0009	.0011	.0014	.0018	.0022	.0027
10	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0006	.0008
11	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002
12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

Appendix Table 11-B

Poisson Distribution (Continued)

x/λ	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
0	.0450	.0408	.0369	.0334	.0302	.0273	.0247	.0224	.0224	.0183
1	.1397	.1304	.1217	.1135	.1057	.0984	.0915	.0850	.0789	.0733
2	.2165	.2087	.2008	.1929	.1850	.1771	.1692	.1615	.1539	.1465
3	.2237	.2226	.2209	.2186	.2158	.2125	.2087	.2046	.2001	.1954
4	.0734	.1781	.1823	.1858	.1888	.1912	.1931	.1944	.1951	.1954
5	.1075	.1140	.1203	.1264	.1322	.1377	.1429	.1477	.1522	.1563
6	.0555	.0608	.0662	.0716	.0771	.0826	.0881	.0936	.0989	.1042
7	.0246	.0278	.0312	.0348	.0385	.0425	.0466	.008	.0551	.1595
8	.0095	.0111	.0129	.0148	.0159	.0191	.0215	.0241	.0269	.0298
9	.0033	.0040	.0047	.0056	.0066	.0076	.0089	.0102	.0116	.0132
10	.0010	.0013	.0016	.0019	.0023	.0028	.0033	.0039	.0045	.0053
11	.0003	.0004	.0005	.0006	.0007	.0009	.0011	.0013	.0016	.0019
12	.0001	.0001	.0001	.0002	.0002	.0003	.0003	.0004	.0005	.0006
13	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002
14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

x/λ	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
0	.0166	.0150	.0136	.0123	.0111	.0101	.0091	.0082	.0074	.0067
1	.0679	.0630	.0583	.0540	.0500	.0462	.0427	.0395	.0365	.0337
2	.1393	.1323	.1254	.1188	.1125	.1063	.1005	.0948	.0894	.0842
3	.1904	.1852	.1798	.1743	.1687	.1631	.1574	.1517	.1460	.1404
4	.1951	.1944	.1933	.1917	.1898	.1875	.1849	.1820	.1789	.1775
5	.1600	.1633	.1662	.1687	.1708	.1725	.1738	.1747	.1753	.1755
6	.1093	.1143	.1191	.1237	.1281	.1323	1362	.1398	.1432	1462
7	.0640	.0686	.0732	.0778	.0824	.0869	.0914	.0959	.1002	.1044
8	.0328	.0360	.0393	.0428	.0463	.0500	.0537	.0537	.0614	.0653
9	.0150	.0168	.0188	.0209	.0232	.0255	.0280	.0307	.0334	.0363
10	.0061	.0071	.0081	.0092	.0104	.0118	.0132	.0147	.0164	.0181
11	.0023	.0027	.0032	.0037	.0043	.0049	.0056	.0064	.0073	.0082
12	.0008	.0009	.0011	.0014	.0016	.0019	.0022	.0026	.0030	.0034
13	.0002	.0003	.0004	.0005	.0006	.0007	.0008	.0009	.0011	.0013
14	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0003	.0004	.0005
15	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001	.0002

Appendix Table 11- C

Poisson Distribution (Continued)

x/λ	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
0	.0061	.0055	.0050	.0045	.0041	.0037	.0033	.0030	.0027	.0025
1	.0311	.0287	.0265	.0244	.0225	.0207	.0191	.0176	.0162	.0149
2	.0793	.0746	.0701	.0659	.0618	.0580	.0544	.0509	.0477	.0446
3	.1348	.1293	.1239	.1185	.1133	.1082	.1033	.0985	.0938	.0892
4	.1719	.1681	.1641	.1600	.1558	.1515	.1472	.1428	.1383	.1339
5	.1753	.1748	.1740	.1728	.1714	.1697	.1678	.1656	.1632	.1606
6	.1490	.1515	.1537	.1555	.1571	.1584	.1594	.1601	.1605	.1606
7	.1086	.1125	.1163	.1200	.1234	.1267	.1298	.1326	.1353	.1377
8	.0692	.0731	.0771	.0810	.0849	.0887	.0925	.0962	.0998	.1033
9	.0362	.0423	.0454	.0486	.0519	.0552	.0586	.0620	.0654	.0688
10	.0200	.0220	.0241	.0262	.0285	.0309	.0334	.0359	.0386	.0413
11	.0093	.0104	.0116	.0129	.0143	.0157	.0173	.0190	.0207	.0225
12	.0039	.0045	.0051	.0058	.0065	.0073	.0082	.0092	.0102	.0113
13	.0015	.0104	.0021	.0024	.0028	.0032	.0036	.0041	.0046	.0052
14	.0006	.0007	.0008	.0009	.0011	.0013	.0015	.0017	.0019	.0022
15	.0002	.0002	.0003	.0003	.0004	.0005	.0006	.0007	.0008	.0009
16	.0001	.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0003	.0003
17	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001

x/λ	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7.0
0	.0022	.0020	.0018	.0017	.0015	.0014	.0012	.0011	.0010	.0000
1	.0137	.0126	.0116	.0106	.0098	.0090	.0082	.0076	.0070	.0064
2	.0417	.0390	.0364	.0340	.0318	.0296	.0276	.0258	.0240	.0223
3	.0848	.0806	.0765	.0726	.0688	.0652	.0617	.0584	.0552	.0521
4	.1294	.1249	.1203	.1162	.1118	.1076	.1034	.0992	.0952	.0912
5	.1579	.1549	.1519	.1487	.1454	.1420	.1385	.1349	.1314	.1277
6	.1605	.1601	.1595	.1586	.1575	.1562	.1546	.1529	.1511	.1490
7	.1399	.1418	.1435	.1450	.1462	.1472	.1480	.1486	.1489	.1490
8	.1066	.1099	.1130	.1160	.1188	.1215	.1240	.1263	.1284	.1304
9	.0723	.0757	.0791	.0825	.0858	.0891	.0923	.0954	.0985	.1014
10	.0441	.0469	.0498	.5285	.0558	.0588	.0618	.0679	.0679	.0710
11	.0245	.0265	.0285	.0307	.0330	.0353	.0377	.0401	.0426	.0452
12	.0124	.0137	.0150	.0164	.0179	.0194	.0210	.0227	.0245	.0264
13	.0058	.0065	.0073	.0081	.0089	.0098	.0108	.0119	.0130	.0142
14	.0025	.0029	.0033	.0037	.0041	.0046	.0052	.0058	.0064	.0071
15	.0010	.0012	.0014	.0016	.0018	.0020	.0023	.0026	.0029	.0033
16	.0004	.0005	.0005	.0006	.0007	.0008	.0010	.0011	.0013	.0014
17	.0001	.0002	.0002	.0002	.0003	.0003	.0004	.0004	.0005	.0006
18	.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0002	.0002	.0002
19	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001

Appendix Table 12

Control Charts

Number of Observations in Sample n	Chart for Averages			Chart for Standard Deviations						Chart for Ranges						
	Factors for Control Limits			Factors for Central Line		Factors for Control Limits				Factors for Central Line		Factors for Control Limits				
	A	A_1	A_2	c_2	$1/c_2$	B_1	B_2	B_3	B_4	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	3.760	1.880	0.5642	1.7725	0	1.843	0	3.267	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	2.394	1.023	0.7236	1.3820	0	1.858	0	2.568	1.693	0.5907	0.888	0	4.358	0	2.576
4	1.500	1.880	0.729	0.7979	1.2533	0	1.808	0	2.266	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	1.596	0.577	0.8407	1.1894	0	1.756	0	2.089	2.326	0.4299	0.864	0	4.918	0	2.115
6	1.225	1.410	0.483	0.8686	1.1512	0.026	1.711	0.030	0.970	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	1.277	0.419	0.8882	1.1259	0.105	1.672	0118	1.882	2.704	0.3698	0.833	0.205	5.203	0.076	1.294
8	1.061	1.175	0.373	0.9027	1.1078	0.167	1.638	0.185	1.815	2.847	0.3512	0.820	0.387	5.307	0.136	1.884
9	1.000	1.094	0.337	0.9139	1.0942	0.219	1.609	0.239	1.761	2.970	0.3367	0.808	0.546	5.394	0.184	1.816
10	0.949	1.028	0.308	0.9227	1.0837	0.262	1.584	0.284	1.716	3.078	0.3249	0.797	0.687	5.489	0.223	1.777
11	0.905	0.973	0.285	0.9300	1.0753	0.299	1.561	0.321	1.679	3.173	0.3152	0.787	0.812	5.534	0.256	1.744
12	0.866	0.925	0.266	0.9359	1.0684	0.331	1.541	0.354	1.646	3.258	0.3059	0.778	0.924	5.592	0.284	1.716
13	0.882	0.884	0.249	0.9410	1.0627	0.359	1.523	0.382	1.618	3.336	0.2998	0.770	1.026	5.546	0.308	1.692
14	0.802	0.848	0.235	0.9453	1.0579	0.384	1.507	0.406	1.594	3.407	0.2935	0.762	1.121	5.523	0.329	1.671
15	0.775	0.816	0.223	0.9490	1.0537	0.406	1.492	0.428	1.572	3.472	0.2880	0.755	1.207	5.737	0.348	1.652
16	0.750	0.788	0.212	0.9523	1.0501	0.427	1.478	0.448	1.552	3.532	0.2831	0.749	1.285	5.779	0.364	1.636
17	0.728	0.762	0.203	0.9551	1.0470	0.445	1.465	0.466	1.534	3.588	0.2787	0.743	1.359	5.817	0.379	1.621
18	0.707	0.738	0.194	0.9576	1.0442	0.461	1.454	0.482	1.518	3.640	0.2747	0.738	1.426	5.854	0.392	1.608
19	0.688	0.717	0.187	0.9599	1.0418	0.477	1.443	0.497	1.503	3.689	0.2711	0.733	1.490	5.888	0.404	1.596
20	0.671	0.697	0.180	0.9619	1.0396	0.491	1.433	0.510	1.490	3.735	0.2677	0.729	1.548	5.922	0.414	1.586
21	0.655	0.679	0.173	0.9638	1.0376	0.504	1.424	0.523	1.477	3.778	0.2647	0.724	1.605	5.950	0.425	1.575
22	0.640	0.662	0.167	0.9655	1.0358	0.516	1.415	0.534	1.466	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.647	0.162	0.9670	1.0342	0.527	1.407	0.545	1.455	3.858	0.2592	0.716	1.710	6.008	0.443	1.557
24	0.612	0.632	0.157	0.9684	1.0327	0.538	1.399	0.555	1.445	3.895	0.2567	0.712	1.759	6.031	0.452	1.548
25	0.600	0.619	0.153	0.9696	1.0313	0.548	1.392	0.565	1.435	3.931	0.2544	0.709	1.804	6.058	0.459	1.541
Over 25	$\frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{n}}$	–	–	–	↑	↑	↑	↑	–	–	–	–	–	–	–

APPENDIX-D

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